# Graph Isomorphism

- $V_1 = V_2 = \{1, 2, \ldots, n\}$.
- Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there exists a permutation $\pi$ on $\{1, 2, \ldots, n\}$ so that $(u, v) \in E_1 \Leftrightarrow (\pi(u), \pi(v)) \in E_2$.
- The task is to answer if $G_1 \cong G_2$.
- No known polynomial-time algorithms.[a]
- The problem is in NP (hence IP).
- It is not likely to be NP-complete.[b]

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[a] The recent bound of Babai (2015) is $2^{O(\log^c n)}$ for some constant $c$.
[b] Schöning (1987).

## GRAPH NONISOMORPHISM

- $V_1 = V_2 = \{1, 2, \ldots, n\}$.
- Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **nonisomorphic** if there exist *no* permutations $\pi$ on $\{1, 2, \ldots, n\}$ so that $(u, v) \in E_1 \Leftrightarrow (\pi(u), \pi(v)) \in E_2$.
- The task is to answer if $G_1 \not\cong G_2$.
- Again, no known polynomial-time algorithms.
  - It is in coNP, but how about NP or BPP?
  - It is not likely to be coNP-complete.[a]
- Surprisingly, GRAPH NONISOMORPHISM $\in$ IP.[b]

[a] Schöning (1987).
[b] Goldreich, Micali, & Wigderson (1986).

# A 2-Round Algorithm

Victor selects a random $i \in \{1, 2\}$;
Victor selects a random permutation $\pi$ on $\{1, 2, \ldots, n\}$;
Victor applies $\pi$ on graph $G_i$ to obtain graph $H$;
Victor sends $(G_1, H)$ to Peggy;
**if** $G_1 \cong H$ **then**
    Peggy sends $j = 1$ to Victor;
**else**
    Peggy sends $j = 2$ to Victor;
**end if**
**if** $j = i$ **then**
    Victor accepts; $\{G_1 \not\cong G_2.\}$
**else**
    Victor rejects; $\{G_1 \cong G_2.\}$
**end if**

# Analysis

- Victor runs in probabilistic polynomial time.
- Suppose $G_1 \ncong G_2$.
  - Peggy is able to tell which $G_i$ is isomorphic to $H$, so $j = i$.
  - So Victor always accepts.
- Suppose $G_1 \cong G_2$.
  - No matter which $i$ is picked by Victor, Peggy or any prover sees 2 *identical* copies.
  - Peggy or any prover with exponential power has only probability one half of guessing $i$ correctly.
  - So Victor erroneously accepts with probability 1/2.
- Repeat the algorithm to obtain the desired probabilities.

## Knowledge in Proofs

- Suppose I know a satisfying assignment to a satisfiable boolean expression.
- I can convince Alice of this by giving her the assignment.
- But then I give her more knowledge than is necessary.
  - Alice can claim that she found the assignment!
  - Login authentication faces essentially the same issue.
  - See `www.wired.com/wired/archive/1.05/atm_pr.html` for a famous ATM fraud in the U.S.

## Knowledge in Proofs (concluded)

- Suppose I always give Alice random bits.
- Alice extracts no knowledge from me by any measure, but I prove nothing.
- Question 1: Can we design a protocol to convince Alice (the knowledge) of a secret without revealing anything extra?
- Question 2: How to define this idea rigorously?

## Zero Knowledge Proofs[a]

An interactive proof protocol $(P, V)$ for language $L$ has the **perfect zero-knowledge** property if:

- For every verifier $V'$, there is an algorithm $M$ with expected polynomial running time.
- $M$ on any input $x \in L$ generates the same probability distribution as the one that can be observed on the communication channel of $(P, V')$ on input $x$.

[a] Goldwasser, Micali, & Rackoff (1985).

## Comments

- Zero knowledge is a property of the prover.
  - It is the robustness of the prover against attempts of the verifier to extract knowledge via interaction.
  - The verifier may deviate arbitrarily (but in polynomial time) from the predetermined program.
  - A verifier cannot use the transcript of the interaction to convince a third-party of the validity of the claim.
  - The proof is hence not transferable.

## Comments (continued)

- Whatever a verifier can "learn" from the specified prover $P$ via the communication channel could as well be computed from the verifier alone.
- The verifier does not learn anything except "$x \in L$."
- Zero-knowledge proofs yield no knowledge in the sense that they can be constructed by the verifier who believes the statement, and yet these proofs do convince him.

## Comments (continued)

- The "paradox" is resolved by noting that it is *not* the transcript of the conversation that convinces the verifier.
- But the fact that this conversation was held "on line."
- *Computational* zero-knowledge proofs are based on complexity assumptions.
  – $M$ only needs to generate a distribution that is computationally indistinguishable from the verifier's view of the interaction.

# Comments (concluded)

- If one-way functions exist, then zero-knowledge proofs exist for every problem in NP.[a]
- If one-way functions exist, then zero-knowledge proofs exist for every problem in PSPACE.[b]
- The verifier can be restricted to the honest one (i.e., it follows the protocol).[c]
- The coins can be public.[d]
- The digital money Zcash (2016) is based on zero-knowledge proofs.

---

[a] Goldreich, Micali, & Wigderson (1986).
[b] Ostrovsky & Wigderson (1993).
[c] Vadhan (2006).
[d] Vadhan (2006).

## Quadratic Residuacity (QR)

- Let $n$ be a product of two distinct primes.
- Assume extracting the square root of a quadratic residue modulo $n$ is hard without knowing the factors.
- QR asks if $x \in Z_n^*$ is a quadratic residues modulo $n$.

## A Useful Corollary of Lemma 82 (p. 687)

**Corollary 83** *Let $n = pq$ be a product of two distinct primes. (1) If $x$ and $y$ are both quadratic residues modulo $n$, then $xy \in Z_n^*$ is a quadratic residue modulo $n$. (2) If $x$ is a quadratic residue modulo $n$ and $y$ is a quadratic nonresidue modulo $n$, then $xy \in Z_n^*$ is a quadratic nonresidue modulo $n$.*

- Suppose $x$ and $y$ are both quadratic residues modulo $n$.
- Let $x \equiv a^2 \bmod n$ and $y \equiv b^2 \bmod n$.
- Now $xy$ is a quadratic residue as $xy \equiv (ab)^2 \bmod n$.

## The Proof (concluded)

- Suppose $x$ is a quadratic residue modulo $n$ and $y$ is a quadratic nonresidue modulo $n$.
- By Lemma 82 (p. 687), $(x \,|\, p) = (x \,|\, q) = 1$ but, say, $(y \,|\, p) = -1$.
- Now $xy$ is a quadratic nonresidue as $(xy \,|\, p) = -1$, again by Lemma 82 (p. 687).

## Zero-Knowledge Proof of QR[a]

Below is a zero-knowledge proof for $x \in Z_n^*$ being a quadratic residue.

1: **for** $m = 1, 2, \ldots, \log_2 n$ **do**
2: Peggy chooses a random $v \in Z_n^*$ and sends $y = v^2 \bmod n$ to Victor;
3: Victor chooses a random bit $i$ and sends it to Peggy;
4: Peggy sends $z = u^i v \bmod n$, where $u$ is a square root of $x$; $\{u^2 \equiv x \bmod n.\}$
5: Victor checks if $z^2 \equiv x^i y \bmod n$;
6: **end for**
7: Victor accepts $x$ if Line 5 is confirmed every time;

[a] Goldwasser, Micali, & Rackoff (1985).

## Analysis

- Suppose $x$ is a quadratic residue.
  - Then $x$'s square root $u$ can be computed by Peggy.
  - Peggy can answer all challenges.
  - Now,

$$z^2 \equiv \left(u^i\right)^2 v^2 \equiv \left(u^2\right)^i v^2 \equiv x^i y \bmod n.$$

  - So Victor will accept $x$.

## Analysis (continued)

- Suppose $x$ is a quadratic nonresidue.
  - Corollary 83 (p. 714) says if $a$ is a quadratic residue, then $xa$ is a quadratic nonresidue.
  - As $y$ is a quadratic residue, $x^i y$ can be a quadratic residue (see Line 5) only when $i = 0$.
  - Peggy can answer only one of the two possible challenges, when $i = 0$.[a]
  - So Peggy will be caught in any given round with probability one half.

[a]Line 5 ($z^2 \equiv x^i y \bmod n$) cannot equate a quadratic residue $z^2$ with a quadratic nonresidue $x^i y$ when $i = 1$.

## Analysis (continued)

- How about the claim of zero knowledge?
- The transcript between Peggy and Victor when $x$ is a quadratic residue can be generated *without* Peggy!
- Here is how.
- Suppose $x$ is a quadratic residue.[a]
- In each round of interaction with Peggy, the transcript is a triplet $(y, i, z)$.
- We present an efficient Bob that generates $(y, i, z)$ with the same probability *without* accessing Peggy's power.

[a]There is no zero-knowledge requirement when $x \notin L$.

# Analysis (concluded)

1: Bob chooses a random $z \in Z_n^*$;

2: Bob chooses a random bit $i$;

3: Bob calculates $y = z^2 x^{-i} \bmod n$;[a]

4: Bob writes $(y, i, z)$ into the transcript;

---

[a]Recall Line 5 on p. 716: Victor checks if $z^2 \equiv x^i y \bmod n$.

## Comments

- Assume $x$ is a quadratic residue.
- For $(y, i, z)$, $y$ is a random quadratic residue, $i$ is a random bit, and $z$ is a random number.
- Bob cheats because $(y, i, z)$ is *not* generated in the same order as in the original transcript.
  - Bob picks Peggy's answer $z$ first.
  - Bob then picks Victor's challenge $i$.
  - Bob finally patches the transcript.

## Comments (concluded)

- So it is not the transcript that convinces Victor, but that *conversation with Peggy is held "on line."*
- The same holds even if the transcript was generated by a cheating Victor's interaction with (honest) Peggy.
- But we skip the details.[a]

[a] Or apply Vadhan (2006).

# Zero-Knowledge Proof of 3 Colorability[a]

```
for i = 1, 2, ..., |E|^2 do
    Peggy chooses a random permutation π of the 3-coloring φ;
    Peggy samples encryption schemes randomly, commits[b] them,
    and sends π(φ(1)), π(φ(2)), ..., π(φ(|V|)) encrypted to Victor;
    Victor chooses at random an edge e ∈ E and sends it to Peggy
    for the coloring of the endpoints of e;
    if e = (u, v) ∈ E then
        Peggy reveals the colors π(φ(u)) and π(φ(v)) and "proves"
        that they correspond to their encryptions;
    else
        Peggy stops;
    end if
```

[a] Goldreich, Micali, & Wigderson (1986).

[b] Contributed by Mr. Ren-Shuo Liu (`D98922016`) on December 22, 2009.

10: **if** the "proof" provided in Line 6 is not valid **then**
11: Victor rejects and stops;
12: **end if**
13: **if** $\pi(\phi(u)) = \pi(\phi(v))$ or $\pi(\phi(u)), \pi(\phi(v)) \not\in \{1, 2, 3\}$ **then**
14: Victor rejects and stops;
15: **end if**
16: **end for**
17: Victor accepts;

## Analysis

- If the graph is 3-colorable and both Peggy and Victor follow the protocol, then Victor always accepts.
- Suppose the graph is not 3-colorable and Victor follows the protocol.
- Let $e$ be an edge that is *not* colored legally.
- Victor will pick it with probability $1/m$ per round, where $m = |\, E \,|$.
- Then however Peggy plays, Victor will reject with probability at least $1/m$ per round.

## Analysis (concluded)

- So Victor will accept with probability at most

$$\left(1 - m^{-1}\right)^{m^2} \leq e^{-m}.$$

- Thus the protocol is a valid IP protocol.
- This protocol yields no knowledge to Victor as all he gets is a bunch of random pairs.
- The proof that the protocol is zero-knowledge to *any* verifier is intricate.[a]

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[a]Or simply cite Vadhan (2006).

## Comments

- Each $\pi(\phi(i))$ is encrypted by a different cryptosystem in Line 3.[a]
  - Otherwise, the coloring will be revealed in Line 6.
- Each edge $e$ must be picked randomly.[b]
  - Otherwise, Peggy will know Victor's game plan and plot accordingly.

---

[a] Contributed by Ms. Yui-Huei Chang (R96922060) on May 22, 2008

[b] Contributed by Mr. Chang-Rong Hung (R96922028) on May 22, 2008

*Approximability*

All science is dominated by
the idea of approximation.
— Bertrand Russell (1872–1970)

Just because the problem is NP-complete
does not mean that
you should not try to solve it.
— Stephen Cook (2002)

## Tackling Intractable Problems

- Many important problems are NP-complete or worse.
- **Heuristics** have been developed to attack them.
- They are **approximation algorithms**.
- How good are the approximations?
  - We are looking for theoretically *guaranteed* bounds, not "empirical" bounds.
- Are there NP problems that cannot be approximated *well* (assuming NP $\neq$ P)?
- Are there NP problems that cannot be approximated *at all* (assuming NP $\neq$ P)?

## Some Definitions

- Given an **optimization problem**, each problem instance $x$ has a set of **feasible solutions** $F(x)$.
- Each feasible solution $s \in F(x)$ has a cost $c(s) \in \mathbb{Z}^+$.
  - Here, cost refers to the quality of the feasible solution, not the time required to obtain it.
  - It is our **objective function**: total distance, number of satisfied clauses, cut size, etc.

## Some Definitions (concluded)

- The **optimum cost** is

$$\text{OPT}(x) = \min_{s \in F(x)} c(s)$$

for a minimization problem.

- It is

$$\text{OPT}(x) = \max_{s \in F(x)} c(s)$$

for a maximization problem.

## Approximation Algorithms

- Let (polynomial-time) algorithm $M$ on $x$ returns a feasible solution.
- $M$ is an $\epsilon$**-approximation algorithm**, where $\epsilon \geq 0$, if for all $x$,

$$\frac{|\, c(M(x)) - \textsc{opt}(x)\,|}{\max(\textsc{opt}(x), c(M(x)))} \leq \epsilon.$$

  – For a minimization problem,

$$\frac{c(M(x)) - \min_{s \in F(x)} c(s)}{c(M(x))} \leq \epsilon.$$

  – For a maximization problem,

$$\frac{\max_{s \in F(x)} c(s) - c(M(x))}{\max_{s \in F(x)} c(s)} \leq \epsilon. \tag{18}$$

## Lower and Upper Bounds

- For a minimization problem,

$$\min_{s \in F(x)} c(s) \leq c(M(x)) \leq \frac{\min_{s \in F(x)} c(s)}{1-\epsilon}.$$

- For a maximization problem,

$$(1-\epsilon) \times \max_{s \in F(x)} c(s) \leq c(M(x)) \leq \max_{s \in F(x)} c(s). \tag{19}$$

# Lower and Upper Bounds (concluded)

- $\epsilon$ ranges between 0 (best) and 1 (worst).
- For minimization problems, an $\epsilon$-approximation algorithm returns solutions within
$$\left[\text{OPT}, \frac{\text{OPT}}{1-\epsilon}\right].$$
- For maximization problems, an $\epsilon$-approximation algorithm returns solutions within
$$[\,(1-\epsilon) \times \text{OPT}, \text{OPT}\,].$$

# Approximation Thresholds

- For each NP-complete optimization problem, we shall be interested in determining the *smallest* $\epsilon$ for which there is a polynomial-time $\epsilon$-approximation algorithm.
- But sometimes $\epsilon$ has no minimum value.
- The **approximation threshold** is the greatest lower bound of all $\epsilon \geq 0$ such that there is a polynomial-time $\epsilon$-approximation algorithm.
- By a standard theorem in real analysis, such a threshold exists.[a]

[a] Bauldry (2009).

## Approximation Thresholds (concluded)

- The approximation threshold of an optimization problem is anywhere between 0 (approximation to any desired degree) and 1 (no approximation is possible).
- If P = NP, then all optimization problems *in NP* have an approximation threshold of 0.
- So assume P $\neq$ NP for the rest of the discussion.

# Approximation Ratio

- $\epsilon$-approximation algorithms can also be measured via the **approximation ratio**:[a]

$$\frac{c(M(x))}{\text{OPT}(x)}.$$

- For a minimization problem, the approximation ratio is

$$1 \leq \frac{c(M(x))}{\min_{s \in F(x)} c(s)} \leq \frac{1}{1-\epsilon}. \tag{20}$$

- For a maximization problem, the approximation ratio is

$$1-\epsilon \leq \frac{c(M(x))}{\max_{s \in F(x)} c(s)} \leq 1. \tag{21}$$

[a] Williamson & Shmoys (2011).

## Approximation Ratio (concluded)

- Suppose there is an approximation algorithm that achieves an approximation ratio of $\theta$.
  - For a minimization problem, it implies a $(1 - \theta^{-1})$-approximation algorithm by Eq. (20).
  - For a maximization problem, it implies a $(1 - \theta)$-approximation algorithm by Eq. (21).

## NODE COVER

- NODE COVER seeks the smallest $C \subseteq V$ in graph $G = (V, E)$ such that for each edge in $E$, at least one of its endpoints is in $C$.
- A heuristic to obtain a good node cover is to iteratively move a node with the *highest degree* to the cover.
- This turns out to produce an approximation ratio of[a]

$$\frac{c(M(x))}{\text{OPT}(x)} = \Theta(\log n).$$

- So it is not an $\epsilon$-approximation algorithm for any constant $\epsilon < 1$ (see p. 740).

[a] Chvátal (1979).

# A 0.5-Approximation Algorithm[a]

$C := \emptyset$;
**while** $E \neq \emptyset$ **do**
    Delete an arbitrary edge $[\,u, v\,]$ from $E$;
    Add $u$ and $v$ to $C$; {Add 2 nodes to $C$ each time.}
    Delete edges incident with $u$ or $v$ from $E$;
**end while**
**return** $C$;

[a]Gavril (1974).

## Analysis

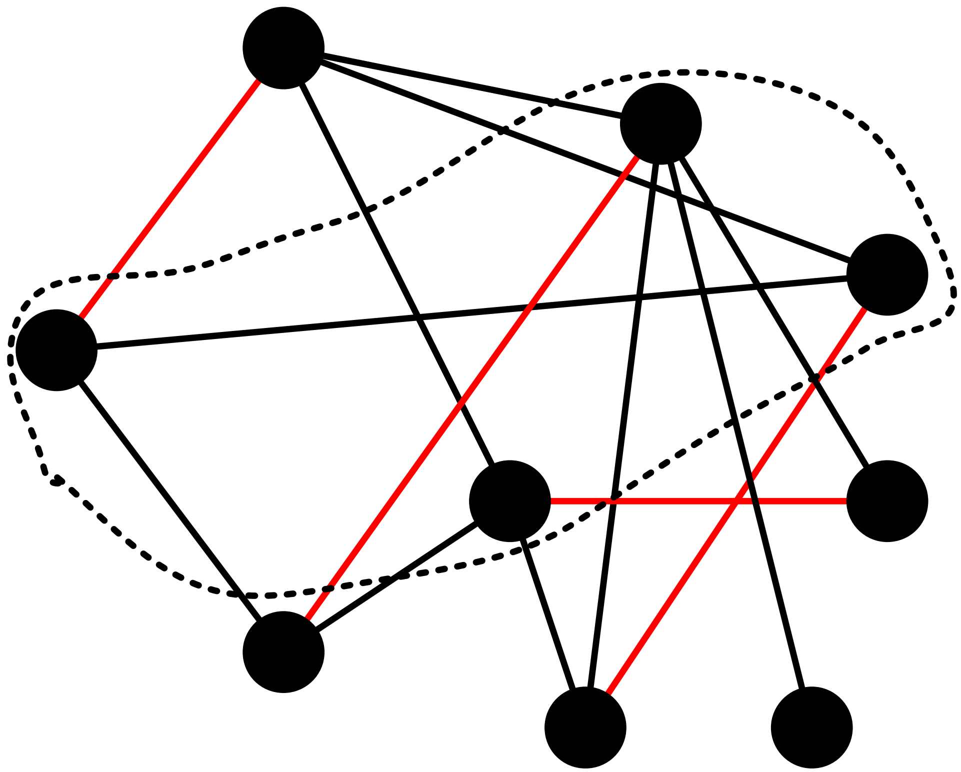
- It is easy to see that $C$ is a node cover.
- $C$ contains $|C|/2$ edges.[a]
- No two edges of $C$ share a node.[b]
- *Any* node cover $C'$ must contain at least one node from *each* of the edges of $C$.
  - If there is an edge in $C$ both of whose ends are outside $C'$, then $C'$ will not be a cover.

---

[a]The edges deleted in Line 3.

[b]In fact, $C$ as a set of edges is a *maximal* matching.

# Analysis (continued)



## Analysis (concluded)

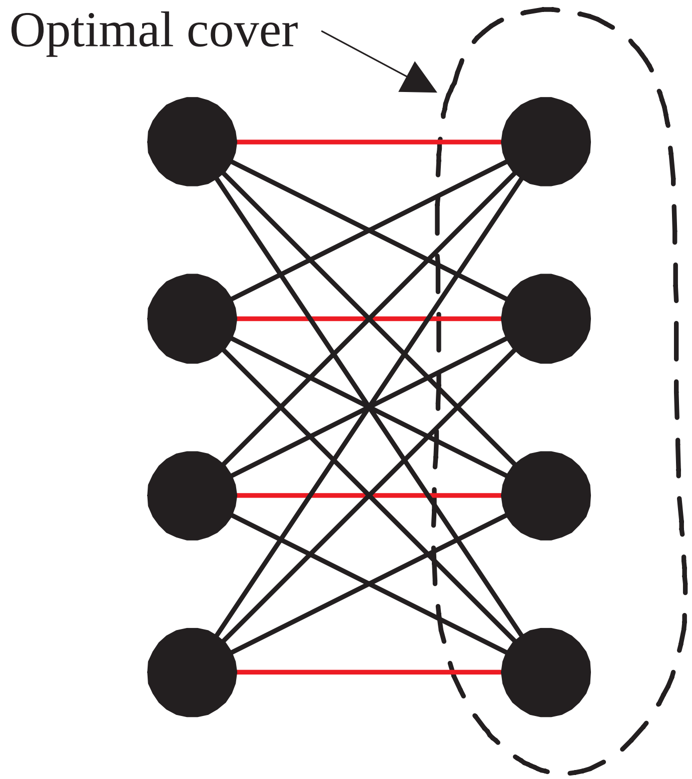
- This means that $\textsc{opt}(G) \geq |\, C \,|/2$.
- The approximation ratio is hence

$$\frac{|\, C \,|}{\textsc{opt}(G)} \leq 2.$$

- So we have a 0.5-approximation algorithm.[a]
- And the approximation threshold is therefore $\leq 0.5$.

[a] Recall p. 740.

# The 0.5 Bound Is Tight for the Algorithm[a]




[a]Contributed by Mr. Jenq-Chung Li (`R92922087`) on December 20, 2003. **König's theorem** says the size of a *maximum* matching equals that of a *minimum* node cover in a bipartite graph.

## Remarks

- The approximation threshold is at least[a]

$$1 - \left(10\sqrt{5} - 21\right)^{-1} \approx 0.2651.$$

- The approximation threshold is 0.5 if one assumes the **unique games conjecture** (UGC).[b]

- This ratio 0.5 is also the lower bound for any "greedy" algorithms.[c]

---

[a] Dinur & Safra (2002).
[b] Khot & Regev (2008).
[c] Davis & Impagliazzo (2004).

## Maximum Satisfiability

- Given a set of clauses, MAXSAT seeks the truth assignment that satisfies the most simultaneously.
- MAX2SAT is already NP-complete (p. 356), so MAXSAT is NP-complete.
- Consider the more general $k$-MAXGSAT for constant $k$.
  - Let $\Phi = \{ \phi_1, \phi_2, \ldots, \phi_m \}$ be a set of boolean expressions in $n$ variables.
  - Each $\phi_i$ is a *general* expression involving up to $k$ variables.
  - $k$-MAXGSAT seeks the truth assignment that satisfies the most expressions simultaneously.

## A Probabilistic Interpretation of an Algorithm

- Let $\phi_i$ involve $k_i \leq k$ variables and be satisfied by $s_i$ of the $2^{k_i}$ truth assignments.
- A random truth assignment $\in \{ 0, 1 \}^n$ satisfies $\phi_i$ with probability $p(\phi_i) = s_i / 2^{k_i}$.
  - $p(\phi_i)$ is easy to calculate as $k$ is a constant.
- Hence a random truth assignment satisfies an average of

$$p(\Phi) = \sum_{i=1}^{m} p(\phi_i)$$

expressions $\phi_i$.

# The Search Procedure

- Clearly
$$p(\Phi) = \frac{p(\Phi[\,x_1 = \texttt{true}\,]) + p(\Phi[\,x_1 = \texttt{false}\,])}{2}.$$
- Select the $t_1 \in \{\,\texttt{true}, \texttt{false}\,\}$ such that $p(\Phi[\,x_1 = t_1\,])$ is the larger one.
- Note that $p(\Phi[\,x_1 = t_1\,]) \geq p(\Phi)$.
- Repeat the procedure with expression $\Phi[\,x_1 = t_1\,]$ until all variables $x_i$ have been given truth values $t_i$ and all $\phi_i$ are either true or false.

## The Search Procedure (continued)

- By our hill-climbing procedure,

$$
\begin{aligned}
& p(\Phi) \\
\leq\ & p(\Phi[\, x_1 = t_1 \,]) \\
\leq\ & p(\Phi[\, x_1 = t_1, x_2 = t_2 \,]) \\
\leq\ & \cdots \\
\leq\ & p(\Phi[\, x_1 = t_1, x_2 = t_2, \ldots, x_n = t_n \,]).
\end{aligned}
$$

- So at least $p(\Phi)$ expressions are satisfied by truth assignment $(t_1, t_2, \ldots, t_n)$.

## The Search Procedure (concluded)

- Note that the algorithm is *deterministic*!
- It is called **the method of conditional expectations**.[a]

[a] Erdős & Selfridge (1973); Spencer (1987).

## Approximation Analysis

- The optimum is at most the number of satisfiable $\phi_i$s—i.e., those with $p(\phi_i) > 0$.
- The ratio of algorithm's output vs. the optimum is[a]

$$\geq \frac{p(\Phi)}{\sum_{p(\phi_i)>0} 1} = \frac{\sum_i p(\phi_i)}{\sum_{p(\phi_i)>0} 1} \geq \min_{p(\phi_i)>0} p(\phi_i).$$

- This is a polynomial-time $\epsilon$-approximation algorithm with $\epsilon = 1 - \min_{p(\phi_i)>0} p(\phi_i)$ by Eq. (21) on p. 739.
- Because $p(\phi_i) \geq 2^{-k}$ for a satisfiable $\phi_i$, the heuristic is a polynomial-time $\epsilon$-approximation algorithm with $\epsilon = 1 - 2^{-k}$.

[a] Because $\sum_i a_i / \sum_i b_i \geq \min_i (a_i / b_i)$.

# Back to MAXSAT

- In MAXSAT, the $\phi_i$'s are clauses (like $x \lor y \lor \neg z$).
- Hence $p(\phi_i) \geq 1/2$ (why?).
- The heuristic becomes a polynomial-time $\epsilon$-approximation algorithm with $\epsilon = 1/2$.[a]
- Suppose we set each boolean variable to true with probability $(\sqrt{5} - 1)/2$, the golden ratio.
- Then follow through the method of conditional expectations to **derandomize** it.

---

[a] Johnson (1974).

## Back to MAXSAT (concluded)

- We will obtain a $[(3-\sqrt{5})]/2$-approximation algorithm.[a]
  - Note $[(3-\sqrt{5})]/2 \approx 0.382$.
- If the clauses have $k$ *distinct* literals,

$$p(\phi_i) = 1 - 2^{-k}.$$

- The heuristic becomes a polynomial-time $\epsilon$-approximation algorithm with $\epsilon = 2^{-k}$.
  - This is the best possible for $k \geq 3$ unless P = NP.
- All the results hold even if clauses are weighted.

[a]Lieberherr & Specker (1981).

## MAX CUT Revisited

- MAX CUT seeks to partition the nodes of graph $G = (V, E)$ into $(S, V - S)$ so that there are as many edges as possible between $S$ and $V - S$.
- It is NP-complete.[a]
- **Local search** starts from a feasible solution and performs "local" improvements until none are possible.
- Next we present a local-search algorithm for MAX CUT.

---

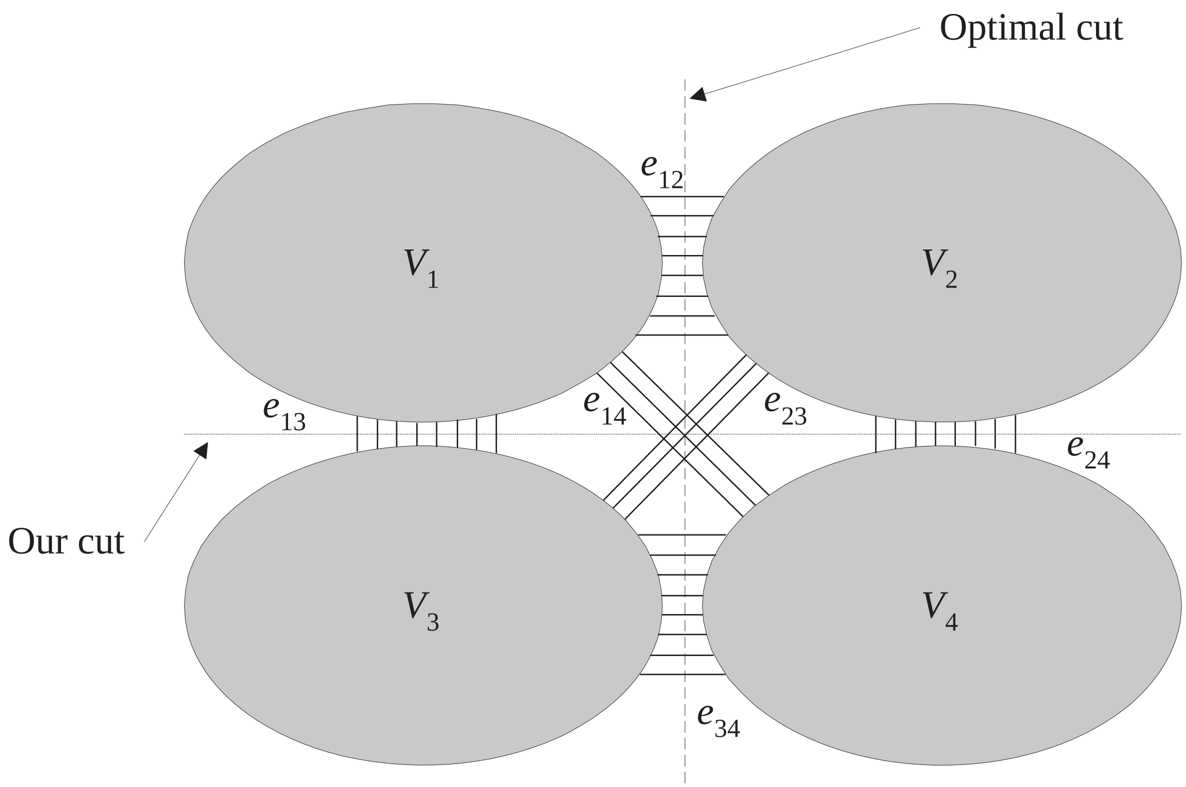
[a]Recall p. 391.

## A 0.5-Approximation Algorithm for MAX CUT

$S := \emptyset$;
**while** $\exists v \in V$ whose switching sides results in a larger cut **do**
  Switch the side of $v$;
**end while**
**return** $S$;

# Analysis



## Analysis (continued)

- Partition $V = V_1 \cup V_2 \cup V_3 \cup V_4$, where
  - Our algorithm returns $(V_1 \cup V_2, V_3 \cup V_4)$.
  - The optimum cut is $(V_1 \cup V_3, V_2 \cup V_4)$.
- Let $e_{ij}$ be the number of edges between $V_i$ and $V_j$.
- Our algorithm returns a cut of size

$$e_{13} + e_{14} + e_{23} + e_{24}.$$

- The optimum cut size is

$$e_{12} + e_{34} + e_{14} + e_{23}.$$

## Analysis (continued)

- For each node $v \in V_1$, its edges to $V_3 \cup V_4$ cannot be outnumbered by those to $V_1 \cup V_2$.
  - Otherwise, $v$ would have been moved to $V_3 \cup V_4$ to improve the cut.
- Considering all nodes in $V_1$ together, we have

$$2e_{11} + e_{12} \leq e_{13} + e_{14}.$$

  - $2e_{11}$, because each edge in $V_1$ is counted twice.
- The above inequality implies

$$e_{12} \leq e_{13} + e_{14}.$$

## Analysis (concluded)

- Similarly,

$$\begin{aligned} e_{12} &\leq e_{23} + e_{24} \\ e_{34} &\leq e_{23} + e_{13} \\ e_{34} &\leq e_{14} + e_{24} \end{aligned}$$

- Add all four inequalities, divide both sides by 2, and add the inequality $e_{14} + e_{23} \leq e_{14} + e_{23} + e_{13} + e_{24}$ to obtain

$$e_{12} + e_{34} + e_{14} + e_{23} \leq 2(e_{13} + e_{14} + e_{23} + e_{24}) = 2 \times \text{OPT}.$$

- The above says our solution is at least half the optimum.

## Remarks

- A 0.12-approximation algorithm exists.[a]
- 0.059-approximation algorithms do not exist unless NP = ZPP.[b]

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[a] Goemans & Williamson (1995).
[b] Håstad (1997).

## Approximability, Unapproximability, and Between

- Some problems have approximation thresholds less than 1.
  - KNAPSACK has a threshold of 0 (p. 778).
  - NODE COVER (p. 745), BIN PACKING, and MAXSAT[a] have a threshold larger than 0.
- The situation is maximally pessimistic for TSP (p. 764) and INDEPENDENT SET,[b] which cannot be approximated
  - Their approximation threshold is 1.

[a] Williamson & Shmoys (2011).
[b] See the textbook.

## Unapproximability of TSP[a]

**Theorem 84** *The approximation threshold of* TSP *is 1 unless* $P = NP$.

- Suppose there is a polynomial-time $\epsilon$-approximation algorithm for TSP for some $\epsilon < 1$.
- We shall construct a polynomial-time algorithm to solve the NP-complete HAMILTONIAN CYCLE.
- Given any graph $G = (V, E)$, construct a TSP with $|V|$ cities with distances

$$d_{ij} = \begin{cases} 1, & \text{if } [i, j] \in E, \\ \frac{|V|}{1-\epsilon}, & \text{otherwise.} \end{cases}$$

[a]Sahni & Gonzales (1976).

## The Proof (continued)

- Run the alleged approximation algorithm on this TSP instance.
- Note that if a tour includes edges of length $|V|/(1-\epsilon)$, then the tour costs more than $|V|$.
- Note also that no tour has a cost less than $|V|$.
- Suppose a tour of cost $|V|$ is returned.
  - Then every edge on the tour exists in the *original* graph $G$.
  - So this tour is a Hamiltonian cycle on $G$.

## The Proof (concluded)

- Suppose a tour that includes an edge of length $|V|/(1-\epsilon)$ is returned.
  - The total length of this tour exceeds $|V|/(1-\epsilon)$.[a]
  - Because the algorithm is $\epsilon$-approximate, the optimum is at least $1-\epsilon$ times the returned tour's length.
  - The optimum tour has a cost exceeding $|V|$.
  - Hence $G$ has no Hamiltonian cycles.

[a] So this reduction is **gap introducing**.

# METRIC TSP

- METRIC TSP is similar to TSP.
- But the distances must satisfy the triangular inequality:

$$d_{ij} \leq d_{ik} + d_{kj}$$

  for all $i, j, k$.
- Inductively,

$$d_{ij} \leq d_{ik} + d_{kl} + \cdots + d_{zj}.$$

## A 0.5-Approximation Algorithm for METRIC TSP[a]

- It suffices to present an algorithm with the approximation ratio of

$$\frac{c(M(x))}{\textsc{opt}(x)} \leq 2$$

(see p. 740).

[a]Choukhmane (1978); Iwainsky, Canuto, Taraszow, & Villa (1986); Kou, Markowsky, & Berman (1981); Plesník (1981).

# A 0.5-Approximation Algorithm for METRIC TSP (concluded)

1: $T :=$ a minimum spanning tree of $G$;

2: $T' :=$ duplicate the edges of $T$ plus their cost; {Note: $T'$ is an Eulerian *multigraph*.}

3: $C :=$ an Euler cycle of $T'$;

4: Remove repeated nodes of $C$; {"Shortcutting."}

5: **return** $C$;

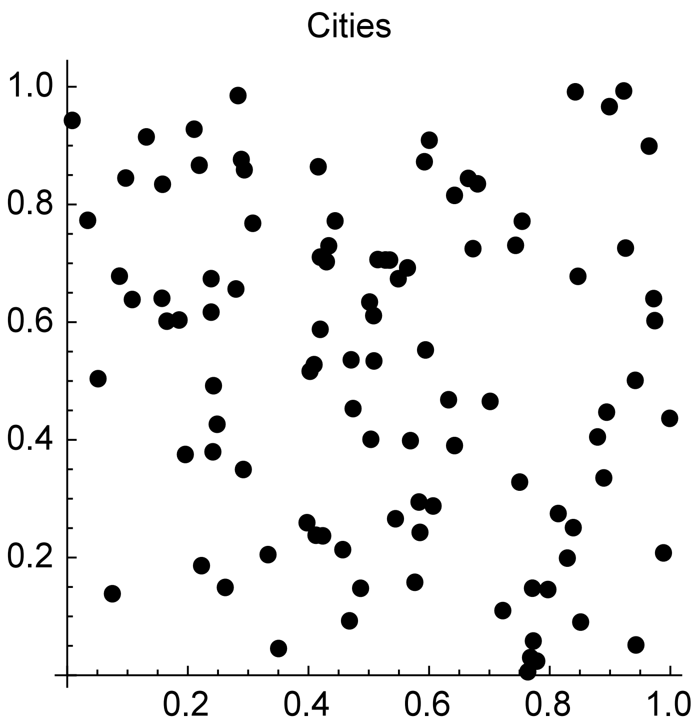
## Analysis

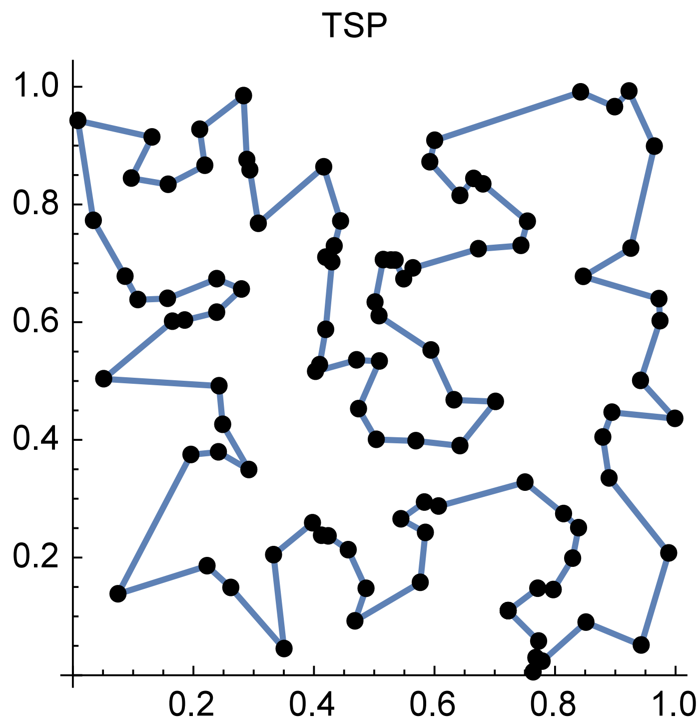
- Let $C_{\text{opt}}$ be an optimal TSP tour.
- Note first that
$$c(T) \leq c(C_{\text{opt}}). \tag{22}$$
  - $C_{\text{opt}}$ is a spanning tree after the removal of one edge.
  - But $T$ is a *minimum* spanning tree.
- Because $T'$ doubles the edges of $T$,
$$c(T') = 2c(T).$$

# Analysis (concluded)

- Because of the triangular inequality, "shortcutting" does not increase the cost.
  - $(1, 2, 3, 2, 1, 4, \ldots) \to (1, 2, 3, 4, \ldots)$, a Hamiltonian cycle.
- Thus
$$c(C) \leq c(T').$$
- Combine all the inequalities to yield
$$c(C) \leq c(T') = 2c(T) \leq 2c(C_{\text{opt}}),$$
as desired.

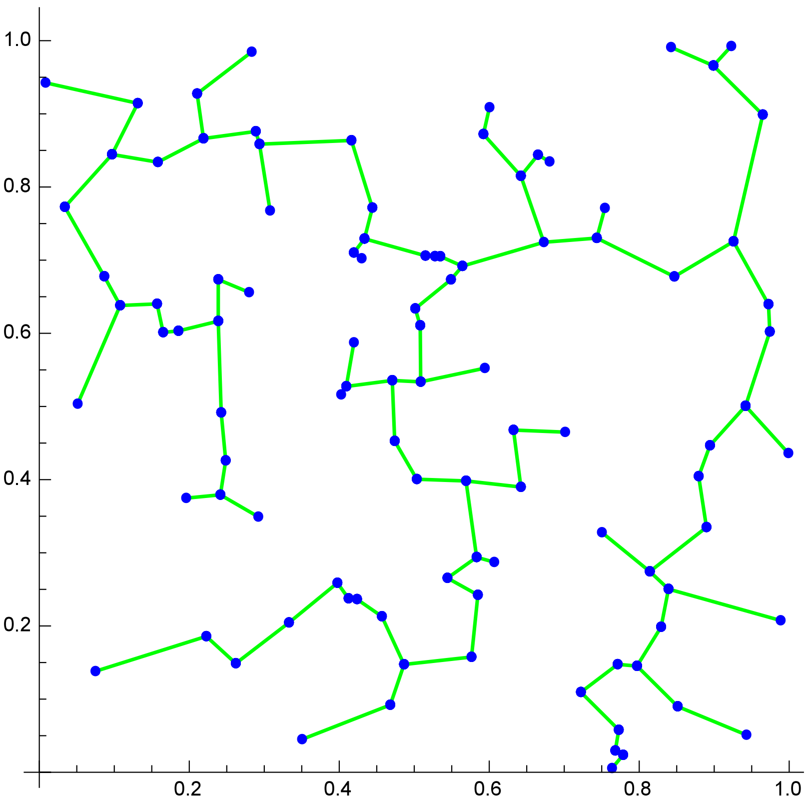
# A 100-Node Example



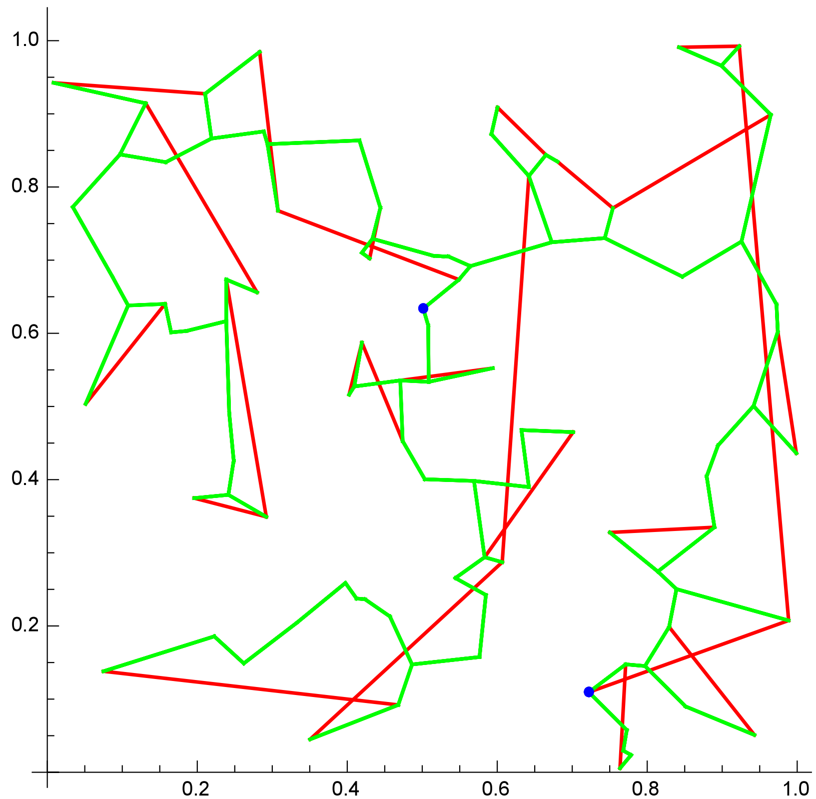

The cost is 7.72877.

# A 100-Node Example (continued)



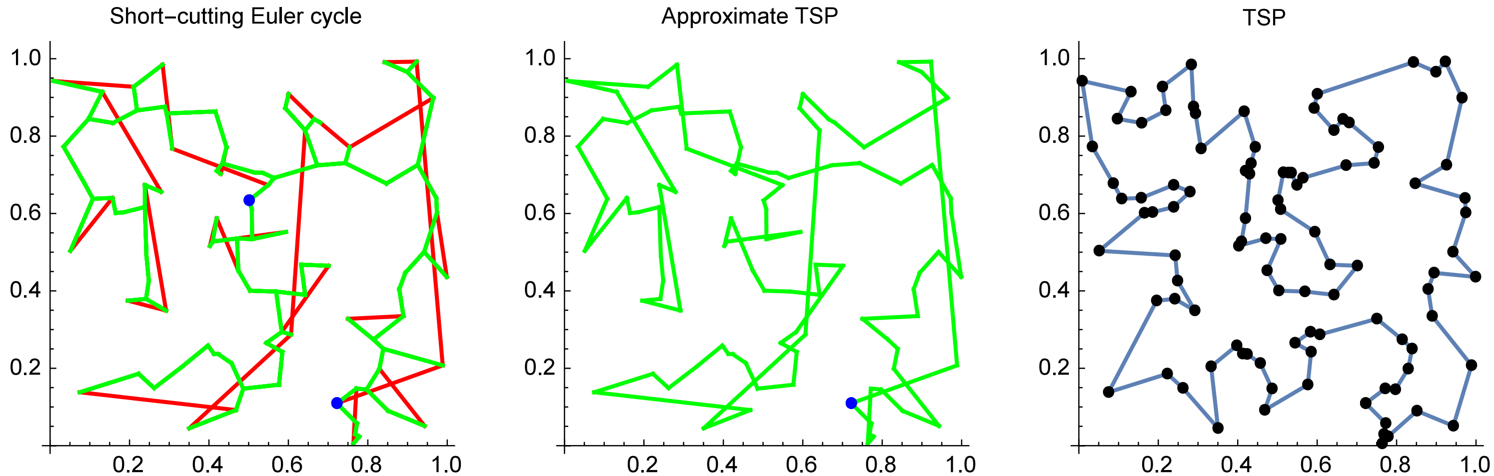

The minimum spanning tree $T$.

# A 100-Node Example (continued)




"Shortcutting" the repeated nodes on the Euler cycle $C$.

# A 100-Node Example (concluded)




The cost is $10.5718 \leq 2 \times 7.72877 = 15.4576$.

## A (1/3)-Approximation Algorithm for METRIC TSP[a]

- It suffices to present an algorithm with the approximation ratio of

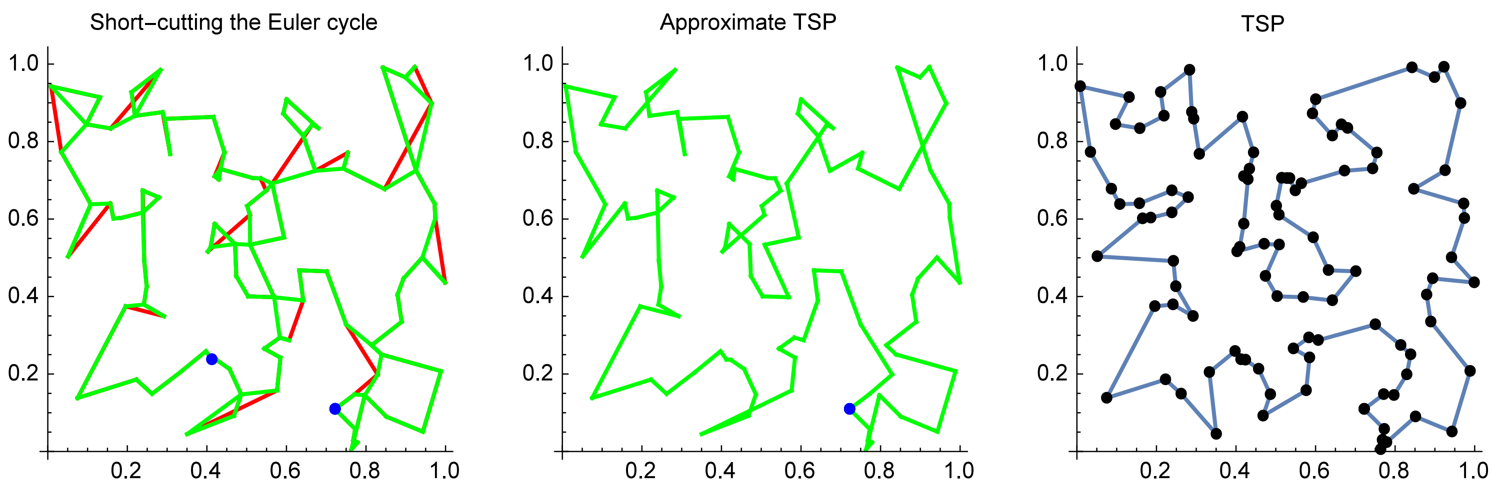
$$\frac{c(M(x))}{\text{OPT}(x)} \leq \frac{3}{2}$$

(see p. 740).

- This is the best approximation ratio for METRIC TSP as of 2016!

[a] Christofides (1976).

# A 100-Node Example[a]




The cost is $8.74583 \leq (3/2) \times 7.72877 = 11.5932$.[b]

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[a] Contributed by Mr. Yu-Chuan Liu (`B00507010`, `R04922040`) on July 15, 2017.

[b] In comparison, the earlier 0.5-approximation algorithm gave a cost of 10.5718 on p. 775.