KNAPSACK

- There is a set of n items.
- Item i has value $v_i \in \mathbb{Z}^+$ and weight $w_i \in \mathbb{Z}^+$.
- We are given $K \in \mathbb{Z}^+$ and $W \in \mathbb{Z}^+$.
- KNAPSACK asks if there exists a subset

$$I \subseteq \{1, 2, \ldots, n\}$$

  such that $\sum_{i \in I} w_i \leq W$ and $\sum_{i \in I} v_i \geq K$.
  - We want to achieve the maximum satisfaction within the budget.

KNAPSACK Is NP-Complete[a]

- KNAPSACK $\in$ NP: Guess an I and check the constraints.
- We shall reduce X3C to KNAPSACK, in which $v_i = w_i$ for all i and $K = W$.
- The simplified KNAPSACK now asks if a subset of $v_1, v_2, \ldots, v_n$ adds up to exactly K.[b]
  - Picture yourself as a radio DJ.

---

[a] Karp (1972). It can be solved in time $O(2^{n/2})$ with space $O(2^{n/4})$ (Schroeppel & Shamir, 1981; Vyskoč, 1987).

[b] This problem is called SUBSET SUM or 0-1 KNAPSACK.

The Proof (continued)

- The primary differences between the two problems are:[a]
  - Sets vs. numbers.
  - Union vs. addition.
- We are given a family $F = \{ S_1, S_2, \ldots, S_n \}$ of size-3 subsets of $U = \{ 1, 2, \ldots, 3m \}$.
- X3C asks if there are m sets in F that cover the set U.
  - These m subsets are disjoint by necessity.

[a]Thanks to a lively class discussion on November 16, 2010.

The Proof (continued)

- Think of a set as a bit vector[a] in $\{0, 1\}^{3m}$.
  - Assume $m = 3$.
  - `110010000` means the set $\{1, 2, 5\}$.
  - `001100010` means the set $\{3, 4, 8\}$.
- Our goal is

$$\overbrace{\mathtt{1}\,\mathtt{1}\cdots\mathtt{1}}^{3m}.$$

[a] Characteristic vector, so to speak.

The Proof (continued)

- A bit vector can also be seen as a binary *number*.
- Set union resembles addition:

$$\begin{array}{rr} & \texttt{001100010} \\ + & \texttt{110010000} \\ \hline & \texttt{111110010} \end{array}$$

which denotes the set $\{1, 2, 3, 4, 5, 8\}$, as desired.

The Proof (continued)

- Trouble occurs when there is *carry*:

$$\begin{array}{rr} & \texttt{010000000} \\ + & \texttt{010000000} \\ \hline & \texttt{100000000} \end{array}$$

  which denotes the wrong set $\{ 1 \}$, not the correct $\{ 2 \}$.

The Proof (continued)

- Or consider

$$\begin{array}{rr} & \texttt{001100010} \\ + & \texttt{001110000} \\ \hline & \texttt{011010010} \end{array}$$

  which denotes the set $\{2, 3, 5, 8\}$, not the correct $\{3, 4, 5, 8\}$.[a]

[a] Corrected by Mr. Chihwei Lin (D97922003) on January 21, 2010.

The Proof (continued)

- Carry may also lead to a situation where we obtain our solution $\overbrace{1\,1\cdots 1}^{3m}$ with more than m sets in F.
- For example, with $m = 3$,

$$
\begin{array}{rr}
 & 000100010 \\
 & 001110000 \\
 & 101100000 \\
+ & 000001101 \\
\hline
 & 111111111
\end{array}
$$

- But the correct union result, $\{1, 3, 4, 5, 6, 7, 8, 9\}$, is *not* an exact cover.

The Proof (continued)

- And it uses 4 sets instead of the required $m = 3$.[a]
- To fix this problem, we enlarge the base just enough so that there are no carries.[b]
- Because there are n vectors in total, we change the base from 2 to $n + 1$.
- Every positive integer N has a unique expression in base b: There are **b-adic digits** $0 \leq d_i < b$ such that

$$N = \sum_{i=0}^{k} d_i b^i, \quad d_k \neq 0.$$

---

[a]Thanks to a lively class discussion on November 20, 2002.

[b]You cannot map $\cup$ to $\vee$ because KNAPSACK requires $+$ not $\vee$!

The Proof (continued)

- Set v_i to be the integer corresponding to the bit vector encoding S_i in base $n+1$:

$$v_i \stackrel{\Delta}{=} \sum_{j \in S_i} 1 \times (n+1)^{3m-j} \tag{4}$$

- Set

$$K \stackrel{\Delta}{=} \sum_{j=0}^{3m-1} 1 \times (n+1)^j = \overbrace{1\,1 \cdots 1}^{3m} \quad \text{(base } n+1\text{)}.$$

- Now in base $n+1$, if there is a set I such that $\sum_{i \in I} v_i = \overbrace{1\,1 \cdots 1}^{3m}$, then every position must be contributed by exactly one v_i and $|\, I \,| = m$.

The Proof (continued)

- For example, the case on p. 437 becomes

$$
\begin{array}{rr}
 & \texttt{000100010} \\
 & \texttt{001110000} \\
 & \texttt{101100000} \\
+ & \texttt{000001101} \\
\hline
 & \texttt{102311111}
\end{array}
$$

in base $n + 1 = 6$.

- As desired, it no longer meets the goal.

The Proof (continued)

- Suppose F admits an exact cover, say $\{ S_1, S_2, \ldots, S_m \}$.
- Then picking $I = \{ 1, 2, \ldots, m \}$ clearly results in

$$v_1 + v_2 + \cdots + v_m = \overbrace{1\,1 \cdots 1}^{3m}.$$

- It is important to note that the meaning of addition (+) is independent of the base.[a]
  - It is just regular addition.
  - But an S_i may give rise to different integers v_i in Eq. (4) on p. 439 under different bases.

[a]Contributed by Mr. Kuan-Yu Chen (R92922047) on November 3, 2004.

The Proof (concluded)

- On the other hand, suppose there exists an I such that

$$\sum_{i \in I} v_i = \overbrace{1\,1 \cdots 1}^{3m}$$

in base $n+1$.

- The no-carry property implies that $|\,I\,| = m$ and

$$\{\, S_i : i \in I \,\}$$

is an exact cover.

SUBSET SUM[a] Is NP-Complete

- The proof actually proves:

Corollary 52 SUBSET SUM *is NP-complete.*

- The proof can be slightly revised to reduce EXACT COVER to SUBSET SUM.
- The proof would *not* work if you used $m+1$ as the base.[b]

---

[a] Recall p. 431.

[b] Contributed by Mr. Yuchen Wang (R08922157) on November 19, 2020.

An Example

- Let $m = 3$, $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and

$$\begin{aligned} S_1 &= \{1, 3, 4\}, \\ S_2 &= \{2, 3, 4\}, \\ S_3 &= \{2, 5, 6\}, \\ S_4 &= \{6, 7, 8\}, \\ S_5 &= \{7, 8, 9\}. \end{aligned}$$

- Note that $n = 5$, as there are 5 S_i's.
- So the base is $n + 1 = 6$.

An Example (continued)

- Our reduction produces

$$
\begin{aligned}
K &= \sum_{j=0}^{3\times 3-1} 6^j = \overbrace{1\,1\cdots 1}^{3\times 3}{}_6 = 2015539_{10},\\
v_1 &= \texttt{101100000}_6 = 1734048_{10},\\
v_2 &= \texttt{011100000}_6 = 334368_{10},\\
v_3 &= \texttt{010011000}_6 = 281448_{10},\\
v_4 &= \texttt{000001110}_6 = 258_{10},\\
v_5 &= \texttt{000000111}_6 = 43_{10}.
\end{aligned}
$$

An Example (concluded)

- Note $v_1 + v_3 + v_5 = K$ because

$$\begin{array}{rr} & \texttt{101100000} \\ & \texttt{010011000} \\ + & \texttt{000000111} \\ \hline & \texttt{111111111} \end{array}$$

- Indeed,

$$S_1 \cup S_3 \cup S_5 = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9 \},$$

an exact cover by 3-sets.

BIN PACKING

- We are given N positive integers $a_1, a_2, \ldots, a_N$, an integer C (the capacity), and an integer B (the number of bins).
- BIN PACKING asks if these numbers can be partitioned into B subsets, each of which has total sum at most C.
- Think of packing bags at the check-out counter.

Theorem 53 BIN PACKING *is NP-complete.*

BIN PACKING (concluded)

- But suppose $a_1, a_2, \ldots, a_N$ are randomly distributed between 0 and 1.
- Let B be the smallest number of unit-capacity bins capable of holding them.
- Then B can deviate from its average by more than t with probability at most $2e^{-2t^2/N}$.[a]

---

[a] Rhee & Talagrand (1987); Dubhashi & Panconesi (2012).

INTEGER PROGRAMMING (IP)

- IP asks whether a system of linear inequalities with integer coefficients has an integer solution.
- In contrast, LINEAR PROGRAMMING (LP) asks whether a system of linear inequalities with integer coefficients has a *rational* solution.
  - LP is solvable in polynomial time.[a]

---

[a]Khachiyan (1979).

IP Is NP-Complete[a]

- SET COVERING can be expressed by the inequalities $Ax \geq \vec{1}$, $\sum_{i=1}^{n} x_i \leq B$, $0 \leq x_i \leq 1$, where
  - $x_i = 1$ if and only if S_i is in the cover.
  - A is the matrix whose columns are the bit vectors of the sets $S_1, S_2, \ldots$.
  - $\vec{1}$ is the vector of 1s.
  - The operations in Ax are standard matrix operations.
  - The ith row of Ax is at least 1 means item i is covered.

[a] Karp (1972); Borosh & Treybig (1976); Papadimitriou (1981).

IP Is NP-Complete (concluded)

- This shows IP is NP-hard.
- Many NP-complete problems can be expressed as an IP problem.
- IP with a fixed number of variables is in P.[a]

[a] Lenstra (1983).

Christos Papadimitriou (1949–)

Easier or Harder?[a]

- Adding restrictions on the allowable *problem instances* will not make a problem harder.
  - We are now solving a subset of problem instances or special cases.
  - The INDEPENDENT SET proof (p. 372) and the KNAPSACK proof (p. 431): equally hard.
  - CIRCUIT VALUE to MONOTONE CIRCUIT VALUE (p. 321): equally hard.
  - SAT to 2SAT (p. 353): easier.

[a] Thanks to a lively class discussion on October 29, 2003.

Easier or Harder? (concluded)

- Adding restrictions on the allowable *solutions* (the solution space) may make a problem harder, equally hard, or easier.
- It is problem dependent.
  - MIN CUT to BISECTION WIDTH (p. 405): harder.
  - LP to IP (p. 449): harder.
  - SAT to NAESAT (p. 365) and MAX CUT to MAX BISECTION (p. 403): equally hard.
  - 3-COLORING to 2-COLORING (p. 415): easier.

coNP and Function Problems

I frankly confess
I do not know what he means.
— St. Augustin (354–430),
City of God (426)

coNP

- By definition, coNP is the class of problems whose complement is in NP.
  - $L \in \text{coNP}$ if and only if $\bar{L} \in \text{NP}$.
- NP problems have succinct certificates.[a]
- coNP is therefore the class of problems that have succinct **disqualifications**:[b]
  - A "no" instance possesses a short proof of its being a "no" instance.
  - Only "no" instances have such proofs.

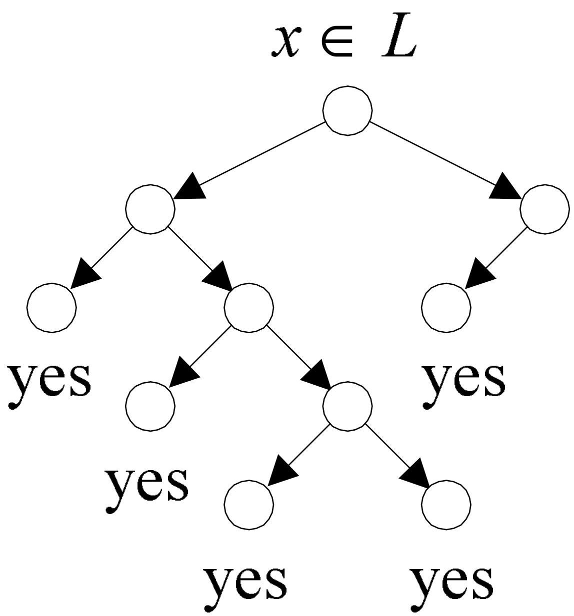
---

[a] Recall Proposition 41 (p. 335).

[b] To be proved in Proposition 54 (p. 466).

coNP (continued)

- Suppose L is a coNP problem.
- There exists a nondeterministic polynomial-time algorithm M such that:
  - If $x \in L$, then $M(x) =$ "yes" for all computation paths.
  - If $x \notin L$, then $M(x) =$ "no" for some computation path.
- If we swap "yes" and "no" in M, the new algorithm decides $\bar{L} \in \text{NP}$ in the classic sense (p. 110).


$x \in L$
yes
yes
yes
yes
yes
$x \notin L$
no
yes
yes
no
yes

coNP (continued)

- So there are 3 major approaches to proving $L \in \text{coNP}$.
  1. Prove $\bar{L} \in \text{NP}$.
     - Especially when you already knew $\bar{L} \in \text{NP}$.
  2. Prove that only "no" instances possess short proofs (for their not being in L).[a]
  3. Write an algorithm for it directly.

---

[a]Recall Proposition 41 (p. 335).

coNP (concluded)

- Clearly $\text{P} \subseteq \text{coNP}$.
- It is not known if

$$\text{P} = \text{NP} \cap \text{coNP}.$$

  - Contrast this with

$$\text{R} = \text{RE} \cap \text{coRE}$$

    (see p. 159).

Some coNP Problems

- SAT COMPLEMENT $\in$ coNP.
  - SAT COMPLEMENT is the complement of SAT.[a]
  - Or, the disqualification is a truth assignment that *satisfies* it.
- HAMILTONIAN PATH COMPLEMENT $\in$ coNP.
  - HAMILTONIAN PATH COMPLEMENT is the complement of HAMILTONIAN PATH.
  - Or, the disqualification is a Hamiltonian path.

[a] Recall p. 200.

Some coNP Problems (concluded)

- VALIDITY $\in$ coNP.
  - If ϕ is not valid, it can be disqualified very succinctly: a truth assignment that does *not* satisfy it.
- TSP COMPLEMENT (D) $\in$ coNP.
  - TSP COMPLEMENT (D) asks if the optimal tour has a total distance of $> B$, where B is an input.[a]
  - The disqualification is a tour with a distance $\leq B$.

---

[a]Defined by Mr. Che-Wei Chang (`R95922093`) on September 27, 2006.

A Nondeterministic Algorithm for SAT COMPLEMENT (See also p. 121)

ϕ is a boolean formula with n variables.

for $i = 1, 2, \ldots, n$ **do**
    Guess $x_i \in \{0, 1\}$; {Nondeterministic choice.}
end for
{Verification:}
if $\phi(x_1, x_2, \ldots, x_n) = 1$ **then**
    "no";
else
    "yes";
end if

Analysis

- The algorithm decides language $\{\,\phi : \phi \text{ is unsatisfiable}\,\}$.
  - The computation tree is a complete binary tree of depth n.
  - Every computation path corresponds to a particular truth assignment out of 2^n.
  - ϕ is unsatisfiable if and only if every truth assignment falsifies ϕ.
  - But every truth assignment falsifies ϕ if and only if every computation path results in "yes."

An Alternative Characterization of coNP

Proposition 54 *Let $L \subseteq \Sigma^*$ be a language. Then $L \in$ coNP if and only if there is a polynomially decidable and polynomially balanced relation R such that*

$$L = \{ x : \forall y \, (x, y) \in R \}.$$

(As on p. 334, we assume $|y| \leq |x|^k$ for some k.)

- $\bar{L} = \{ x : \exists y \, (x, y) \in \neg R \}$.[a]
- Because $\neg R$ remains polynomially balanced, $\bar{L} \in$ NP by Proposition 41 (p. 335).
- Hence $L \in$ coNP by definition.

---

[a]So a certificate y for $\bar{L}$ *is* a disqualification for L, and vice versa.

coNP-Completeness

Proposition 55 *L is NP-complete if and only if its complement $\bar{L} = \Sigma^* - L$ is coNP-complete.*

Proof ($\Rightarrow$; the $\Leftarrow$ part is symmetric)

- Let $\overline{L'}$ be any coNP language.
- Hence $L' \in$ NP.
- Let R be the reduction from L' to L.
- So $x \in L'$ if and only if $R(x) \in L$.
- By the law of transposition, $x \notin L'$ if and only if $R(x) \notin L$.

coNP Completeness (concluded)

- So $x \in \overline{L'}$ if and only if $R(x) \in \bar{L}$.
- The *same* R is a reduction from $\overline{L'}$ to $\bar{L}$.
- This shows $\bar{L}$ is coNP-hard.
- But $\bar{L} \in$ coNP.
- This shows $\bar{L}$ is coNP-complete.

Some coNP-Complete Problems

- SAT COMPLEMENT is coNP-complete.
- HAMILTONIAN PATH COMPLEMENT is coNP-complete.
- TSP COMPLEMENT (D) is coNP-complete.
- VALIDITY is coNP-complete.
  - ϕ is valid if and only if $\neg\phi$ is not satisfiable.
  - $\phi \in$ VALIDITY if and only if $\neg\phi \in$ SAT COMPLEMENT.
  - The reduction from SAT COMPLEMENT to VALIDITY is hence easy: $R(\phi) = \neg\phi$.

Possible Relations between P, NP, coNP

1. $P = NP = coNP$.
2. $NP = coNP$ but $P \neq NP$.
3. $NP \neq coNP$ and $P \neq NP$.
   - This is the current "consensus."[a]

[a] Carl Gauss (1777–1855), "I could easily lay down a multitude of such propositions, which one could neither prove nor dispose of."

The Primality Problem

- An integer p is **prime** if $p > 1$ and all positive numbers other than 1 and p itself cannot divide it.
- PRIMES asks if an integer N is a prime number.
- Dividing N by $2, 3, \ldots, \sqrt{N}$ is *not* efficient.
  - The length of N is only $\log N$, but $\sqrt{N} = 2^{0.5 \log N}$.
  - It is an exponential-time algorithm.
- A polynomial-time algorithm for PRIMES was not found until 2002 by Agrawal, Kayal, and Saxena!
- The running time is $\tilde{O}(\log^{7.5} N)$.

```
if n = a^b for some a, b > 1 then
  return "composite";
end if
for r = 2, 3, ..., n − 1 do
  if gcd(n, r) > 1 then
    return "composite";
  end if
  if r is a prime then
    Let q be the largest prime factor of r − 1;
    if q ≥ 4√r log n and n^((r−1)/q) ≠ 1 mod r then
      break; {Exit the for-loop.}
    end if
  end if
end for{r − 1 has a prime factor q ≥ 4√r log n.}
for a = 1, 2, ..., 2√r log n do
  if (x − a)^n ≠ (x^n − a) mod (x^r − 1) in Z_n[ x ] then
    return "composite";
  end if
end for
return "prime"; {The only place with "prime" output.}
```

The Primality Problem (concluded)

- Later, we will focus on efficient "randomized" algorithms for PRIMES (used in *Mathematica*, e.g.).
- $\text{NP} \cap \text{coNP}$ is the class of problems that have succinct certificates *and* succinct disqualifications.
  - Each "yes" instance has a succinct certificate.
  - Each "no" instance has a succinct disqualification.
  - No instances have both.
- We will see that PRIMES $\in \text{NP} \cap \text{coNP}$.
  - In fact, PRIMES $\in$ P as mentioned earlier.

Basic Modular Arithmetics[a]

- Let $m, n \in \mathbb{Z}^+$.
- $m \mid n$ means m divides n; m is n's **divisor**.
- We call the numbers $0, 1, \ldots, n-1$ the **residue** modulo n.
- The **greatest common divisor** of m and n is denoted $\gcd(m, n)$.

[a] Carl Friedrich Gauss.

Basic Modular Arithmetics (concluded)

- We use
$$a \equiv b \mod n$$
if $n \mid (a - b)$.
  - So $25 \equiv 38 \bmod 13$.
- We use
$$a = b \bmod n$$
if b is the remainder of a divided by n.
  - So $25 = 12 \bmod 13$.

Primitive Roots in Finite Fields

Theorem 56 (Lucas & Lehmer, 1927) [a] *A number* $p > 1$ *is a prime if and only if there is a number* $1 < r < p$ *such that*

1. $r^{p-1} = 1 \bmod p$, *and*
2. $r^{(p-1)/q} \neq 1 \bmod p$ *for all prime divisors* q *of* $p - 1$.

- This r is called a **primitive root** or **generator** of p.
- We will prove one direction of the theorem later.[b]

---

[a] François Edouard Anatole Lucas (1842–1891); Derrick Henry Lehmer (1905–1991).

[b] See pp. 487ff.

Derrick Lehmer[a] (1905–1991)

[a]Inventor of the linear congruential generator in 1951.

Pratt's Theorem

Theorem 57 (Pratt, 1975) PRIMES $\in$ *NP* $\cap$ *coNP.*

- PRIMES $\in$ coNP because a succinct disqualification is a proper divisor.
  - A proper divisor of a number means it is *not* a prime.
- Now suppose p is a prime.
- p's certificate includes the r in Theorem 56 (p. 476).
  - There may be multiple choices for r.

The Proof (continued)

- Use recursive doubling to check if $r^{p-1} = 1 \bmod p$ in time polynomial in the length of the input, $\log_2 p$.
  - $r, r^2, r^4, \ldots \bmod p$, a total of $\sim \log_2 p$ steps.
- We also need all *prime* divisors of $p - 1$: $q_1, q_2, \ldots, q_k$.
  - Whether $r, q_1, \ldots, q_k$ are easy to find is irrelevant.
- Checking $r^{(p-1)/q_i} \neq 1 \bmod p$ is also easy.
- Checking $q_1, q_2, \ldots, q_k$ are all the divisors of $p - 1$ is easy.

The Proof (concluded)

- We still need certificates for the primality of the q_i's.
- The complete certificate is recursive and tree-like:

$$C(p) = (r; q_1, C(q_1), q_2, C(q_2), \ldots, q_k, C(q_k)). \quad (5)$$

- We next prove that $C(p)$ is succinct.
- As a result, $C(p)$ can be checked in polynomial time.

A Certificate for 23[a]

- Note that 5 is a primitive root modulo 23 and $23 - 1 = 22 = 2 \times 11$.[b]
- So
$$C(23) = (5; 2, C(2), 11, C(11)).$$
- Note that 2 is a primitive root modulo 11 and $11 - 1 = 10 = 2 \times 5$.
- So
$$C(11) = (2; 2, C(2), 5, C(5)).$$

[a] Thanks to a lively discussion on April 24, 2008.

[b] Other primitive roots are $7, 10, 11, 14, 15, 17, 19, 20, 21$.

A Certificate for 23 (concluded)

- Note that 2 is a primitive root modulo 5 and $5 - 1 = 4 = 2^2$.
- So
$$C(5) = (2; 2, C(2)).$$
- In summary,
$$C(23) = (5; 2, C(2), 11, (2; 2, C(2), 5, (2; 2, C(2)))).$$
  - In *Mathematica*, `PrimeQCertificate[23]` yields
$$\{ 23, 5, \{ 2, \{ 11, 2, \{ 2, \{ 5, 2, \{ 2 \}\}\}\}\}\}$$

The Succinctness of the Certificate

Lemma 58 *The length of $C(p)$ is at most quadratic at* $5\log_2^2 p$.

- This claim holds when $p = 2$ or $p = 3$.
- In general, $p - 1$ has $k \leq \log_2 p$ prime divisors $q_1 = 2, q_2, \ldots, q_k$.
  - Reason:

$$2^k \leq \prod_{i=1}^{k} q_i \leq p - 1.$$

- Note also that, as $q_1 = 2$,

$$\prod_{i=2}^{k} q_i \leq \frac{p-1}{2}. \tag{6}$$

The Proof (continued)

- $C(p)$ requires:
  - 2 parentheses;
  - $2k < 2\log_2 p$ separators (at most $2\log_2 p$ bits);
  - r (at most $\log_2 p$ bits);
  - $q_1 = 2$ and its certificate 1 (at most 5 bits);
  - $q_2, \ldots, q_k$ (at most $2\log_2 p$ bits);[a]
  - $C(q_2), \ldots, C(q_k)$.

[a]Why?

The Proof (concluded)

- $C(p)$ is succinct because, by induction,

$$
\begin{aligned}
|\, C(p)\, | &\leq 5\log_2 p + 5 + 5\sum_{i=2}^{k} \log_2^2 q_i \\
&\leq 5\log_2 p + 5 + 5\left(\sum_{i=2}^{k} \log_2 q_i\right)^2 \\
&\leq 5\log_2 p + 5 + 5\log_2^2 \frac{p-1}{2} \quad \text{by inequality (6)} \\
&< 5\log_2 p + 5 + 5[\,(\log_2 p) - 1\,]^2 \\
&= 5\log_2^2 p + 10 - 5\log_2 p \leq 5\log_2^2 p
\end{aligned}
$$

for $p \geq 4$.

Turning the Proof into an Algorithm[a]

- How to turn the proof into a nondeterministic polynomial-time algorithm?
- First, guess a $\log_2 p$-bit number r.
- Then guess up to $\log_2 p$ numbers $q_1, q_2, \ldots, q_k$ each containing at most $\log_2 p$ bits.
- Then recursively do the same thing for each of the q_i to form a certificate (5) on p. 480.
- Finally check if the two conditions of Theorem 56 (p. 476) hold throughout the tree.

---

[a]Contributed by Mr. Kai-Yuan Hou (B99201038, R03922014) on November 24, 2015.

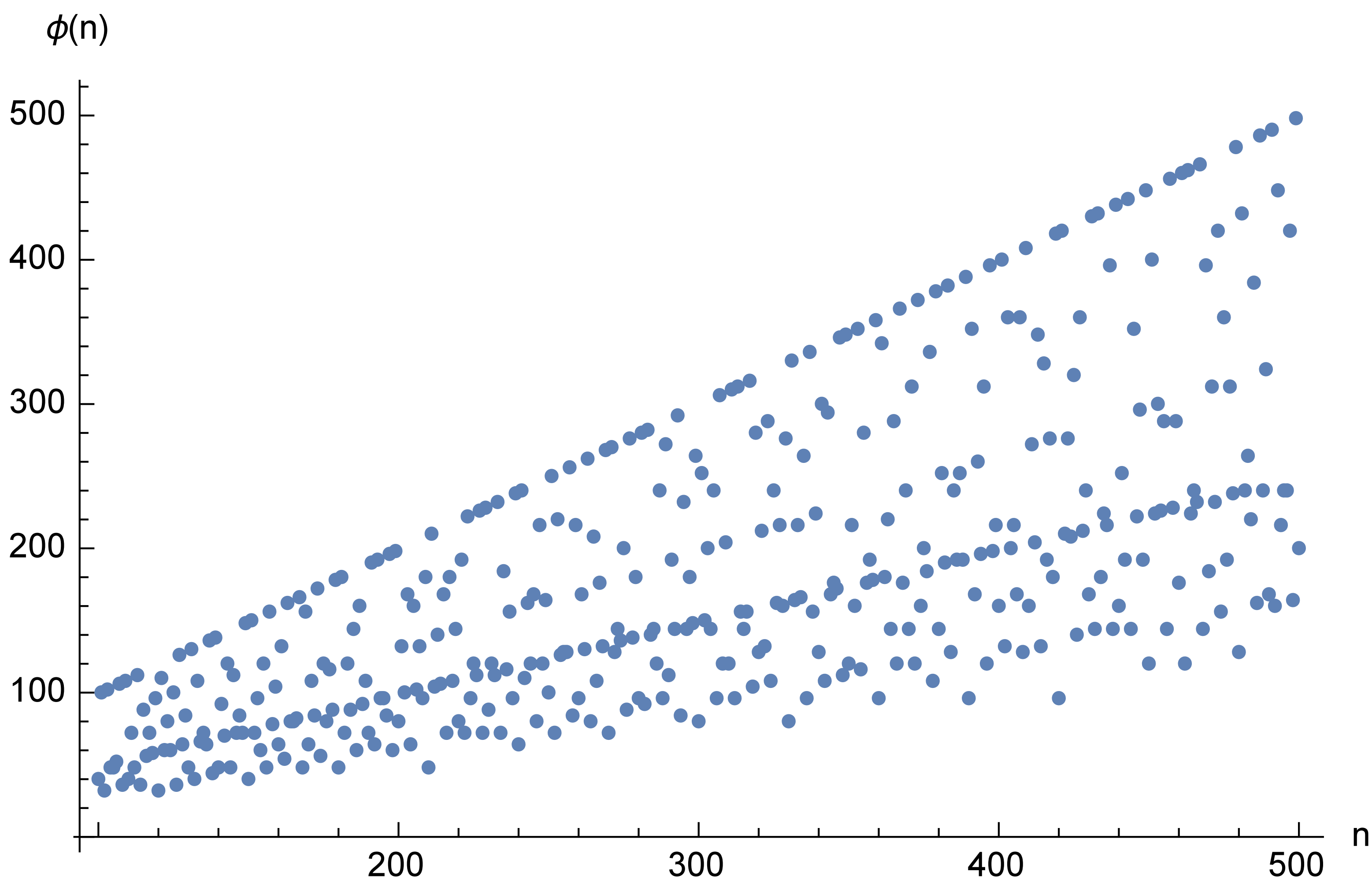
Euler's[a] Totient or Phi Function

- Let
$$\Phi(n) = \{\, m : 1 \leq m < n, \gcd(m, n) = 1 \,\}$$
be the set of all positive integers less than n that are prime to n.[b]
  - $\Phi(12) = \{\, 1, 5, 7, 11 \,\}$.
- Define **Euler's function** of n to be $\phi(n) = |\, \Phi(n) \,|$.
- $\phi(p) = p - 1$ for prime p, and $\phi(1) = 1$ by convention.
- Euler's function is not expected to be easy to compute without knowing n's factorization.

---

[a] Leonhard Euler (1707–1783).

[b] Z_n^* is an alternative notation.



ϕ(n)
500
400
300
200
100
200
300
400
500
n

Leonhard Euler (1707–1783)

Three Properties of Euler's Function[a]

The inclusion-exclusion principle[b] can be used to prove the following.

Lemma 59 *If* $n = p_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell}$ *is the prime factorization of* n*, then*

$$\phi(n) = n \prod_{i=1}^{\ell} \left(1 - \frac{1}{p_i}\right).$$

- For example, if $n = pq$, where p and q are distinct primes, then

$$\phi(n) = pq \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{q}\right) = pq - p - q + 1.$$

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[a] See p. 224 of the textbook.

[b] Consult any textbooks on discrete mathematics.

Three Properties of Euler's Function (concluded)

Corollary 60 $\phi(mn) = \phi(m)\,\phi(n)$ *if* $\gcd(m, n) = 1$.

Lemma 61 (Gauss) $\sum_{m|n} \phi(m) = n$.

The Chinese Remainder Theorem

- Let $n = n_1 n_2 \cdots n_k$, where n_i are pairwise relatively prime.
- For any integers $a_1, a_2, \ldots, a_k$, the set of simultaneous equations

$$
\begin{aligned}
x &= a_1 \bmod n_1, \\
x &= a_2 \bmod n_2, \\
&\vdots \\
x &= a_k \bmod n_k,
\end{aligned}
$$

has a unique solution modulo n for the unknown x.

Fermat's "Little" Theorem[a]

Lemma 62 *For all* $0 < a < p$, $a^{p-1} = 1 \bmod p$.

- Recall $\Phi(p) = \{ 1, 2, \ldots, p-1 \}$.
- Consider $a\Phi(p) = \{ am \bmod p : m \in \Phi(p) \}$.
- $a\Phi(p) = \Phi(p)$.
  - $a\Phi(p) \subseteq \Phi(p)$ as a remainder must be between 1 and $p-1$.
  - Suppose $am \equiv am' \bmod p$ for $m > m'$, where $m, m' \in \Phi(p)$.
  - That means $a(m-m') = 0 \bmod p$, and p divides a or $m-m'$, which is impossible.

[a] Pierre de Fermat (1601–1665).

The Proof (concluded)

- Multiply all the numbers in $\Phi(p)$ to yield $(p-1)!$.
- Multiply all the numbers in $a\Phi(p)$ to yield $a^{p-1}(p-1)!$.
- As $a\Phi(p) = \Phi(p)$, we have

$$a^{p-1}(p-1)! \equiv (p-1)! \bmod p.$$

- Finally, $a^{p-1} = 1 \bmod p$ because $p \not| (p-1)!$.

The Fermat-Euler Theorem[a]

Corollary 63 *For all $a \in \Phi(n)$, $a^{\phi(n)} = 1 \bmod n$.*

- The proof is similar to that of Lemma 62 (p. 493).
- Consider $a\Phi(n) = \{ am \bmod n : m \in \Phi(n) \}$.
- $a\Phi(n) = \Phi(n)$.
  - $a\Phi(n) \subseteq \Phi(n)$ as a remainder must be between 0 and $n - 1$ and relatively prime to n.
  - Suppose $am \equiv am' \bmod n$ for $m' < m < n$, where $m, m' \in \Phi(n)$.
  - That means $a(m - m') = 0 \bmod n$, and n divides a or $m - m'$, which is impossible.

[a]Proof by Mr. Wei-Cheng Cheng (`R93922108`, `D95922011`) on November 24, 2004.

The Proof (concluded)[a]

- Multiply all the numbers in $\Phi(n)$ to yield $\prod_{m \in \Phi(n)} m$.
- Multiply all the numbers in $a\Phi(n)$ to yield $a^{\phi(n)} \prod_{m \in \Phi(n)} m$.
- As $a\Phi(n) = \Phi(n)$,

$$\prod_{m \in \Phi(n)} m \equiv a^{\phi(n)} \left(\prod_{m \in \Phi(n)} m \right) \bmod n.$$

- Finally, $a^{\phi(n)} = 1 \bmod n$ because $n \not| \prod_{m \in \Phi(n)} m$.

[a] Some typographical errors corrected by Mr. Jung-Ying Chen (D95723006) on November 18, 2008.

An Example

- As $12 = 2^2 \times 3$,

$$\phi(12) = 12 \times \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 4.$$

- In fact, $\Phi(12) = \{ 1, 5, 7, 11 \}$.
- For example,

$$5^4 = 625 = 1 \bmod 12.$$