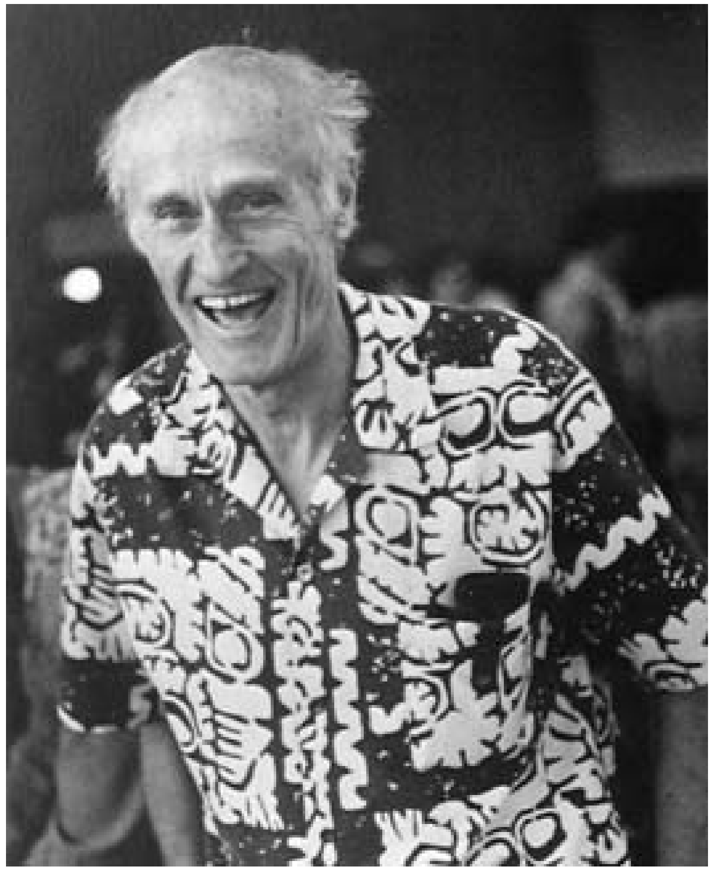
The Kleene Star[a] $*$

- Let A be a set.
- The **Kleene star** of A, denoted by A^*, is the set of all strings obtained by concatenating zero or more strings from A.
  - For example, suppose $A = \{ 0, 1 \}$.
  - Then
  $$A^* = \{ \epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots \}.$$
  - Note that every string in A^* must be of finite length.

[a] Kleene (1956).

Stephen Kleene (1909–1994)



The two words in the language I most respect
are Yes and No.
— Henry James (1843–1916),
The Portrait of a Lady (1881)

Decidability and Recursive Languages

- Let $L \subseteq (\Sigma - \{\sqcup\})^*$ be a **language**, i.e., a set of strings of non-$\sqcup$ symbols, with a *finite* length.
  - For example, $\{0, 01, 10, 210, 1010, \ldots\}$.
- Let M be a TM such that for any string x:
  - If $x \in L$, then $M(x) =$ "yes."
  - If $x \notin L$, then $M(x) =$ "no."
- We say M **decides** L.
- If there exists a TM that decides L, then L is said to be **recursive**[a] or **decidable**.

[a]Little to do with the concept of "recursive" calls.

Recursive and Nonrecursive Languages: Examples

- The set of palindromes over any alphabet is recursive.[a]
  - PALINDROME cannot be solved by finite state automata.
  - In fact, finite-state automata are equivalent to read-only, right-moving TMs.[b]
- The set of prime numbers $\{2, 3, 5, 7, 11, 13, 17, \ldots\}$ is recursive.[c]

---

[a] There is a program that will halt and it returns "yes" if and only if the input is a palindrome.

[b] Thanks to a lively discussion on September 15, 2015.

[c] There is a program that will halt and it returns "yes" if and only if the input is a prime.

Recursive and Nonrecursive Languages: Examples (concluded)

- The set of C programs that do not contain a `while`, a `for`, or a `goto` is recursive.[a]
- But, the set of C programs that do not contain an infinite loop is *not* recursive (see p. 140).[b]

---

[a]There is a program that will halt and it returns "yes" if and only if the input C code does not contain any of the keywords.

[b]So there is no algorithm that will answer correctly in a finite amount of time if a C program will run into an infinite loop on some inputs.

Acceptability and Recursively Enumerable Languages

- Let $L \subseteq (\Sigma - \{\sqcup\})^*$ be a language.
- Let M be a TM such that for any string x:
  - If $x \in L$, then $M(x) =$ "yes."
  - If $x \notin L$, then $M(x) = \nearrow$.[a]
- We say M **accepts** L.
- If L is accepted by some TM, then L is said to be **recursively enumerable** or **semidecidable**.[b]

[a] This part is different from recursive languages.
[b] Post (1944).

Acceptability and Recursively Enumerable Languages (concluded)

- A recursively enumerable language can be *generated* by a TM, thus the name.[a]
  - It means there is a program such that every $x \in L$ (and only they) will be printed out eventually.
- Of course, if L is infinite in size, this program will not terminate.

[a] Proposition 3.5 on p. 61 of the textbook proves it. Thanks to lively class discussions on September 20, 2011, and September 12, 2017.

Emil Post (1897–1954)

W. V. Quine (1985), "E. L. Post worked alone in New York, little heeded."

Recursive and Recursively Enumerable Languages

Proposition 2 *If L is recursive, then it is recursively enumerable.*

- Let TM M decide L.
- Need to design a TM that accepts L.
- We will modify M to obtain an M' that accepts L.

The Proof (concluded)

- M' is identical to M except that when M is about to halt with a "no" state, M' goes into an infinite loop.
  - Simply replace every instruction that results in a "no" state with ones that move the cursor to the right forever and never halts.
- M' accepts L.
  - If $x \in L$, then $M'(x) = M(x) =$ "yes."
  - If $x \notin L$, then $M(x) =$ "no" and so $M'(x) = \nearrow$.

Recursively Enumerable Languages: Examples

- The set of C program-input pairs that do *not* run into an infinite loop is recursively enumerable.
  - Just run its binary code in a simulator environment.
  - Then the simulator will terminate if and only if the C program will terminate.
  - When the C program terminates, the simulator simply exits with a "yes" state.
- The set of C programs that contain an infinite loop is *not* recursively enumerable.[a]

---

[a] See p. 160 for the proof.

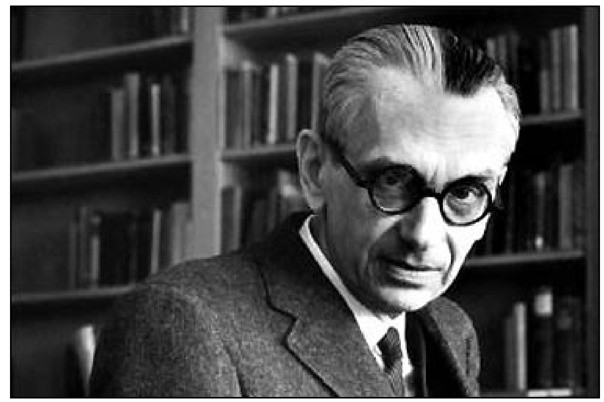
Turing-Computable Functions

- Let $f : (\Sigma - \{\sqcup\})^* \to \Sigma^*$.
  - Optimization problems, root finding problems, etc.
- Let M be a TM with alphabet Σ.
- M **computes** f if for any string $x \in (\Sigma - \{\sqcup\})^*$, $M(x) = f(x)$.
  - f may be a *partial* function.
  - Then $f(x)$ is undefined if and only if $M(x)$ diverges.
- We call f a **(partial) recursive function**[a] if such an M exists.

[a] Gödel (1931, 1934); Kleene (1936).

Kurt Gödel[a] (1906–1978)

Quine (1978), "this theorem [$\cdots$] sealed his immortality."



[a]This photo was taken by Alfred Eisenstaedt (1898–1995).

Church's Thesis

- What is computable is Turing-computable; TMs are algorithms.[a]
- No "intuitively computable" problems have been shown not to be Turing-computable (yet).[b]

[a] Church (1936); Kleene (1943, 1953).

[b] Quantum computer of Manin (1980) and Feynman (1982); DNA computer of Adleman (1994).

Church's Thesis (continued)

- Many other computation models have been proposed.
  - **Recursive function**,[a] λ calculus,[b] **boolean circuits**,[c] **formal language**,[d] **assembly language-like RAM**,[e] **cellular automaton**,[f] **recurrent neural network**,[g] and extensions of the Turing machine (more strings, two-dimensional strings, etc.).

---

[a] Skolem (1923); Gödel (1934); Kleene (1936).
[b] Church (1936).
[c] Shannon (1937).
[d] Post (1943).
[e] Shepherdson & Sturgis (1963).
[f] Conway (1970).
[g] Siegelmann & Sontag (1991).

Church's Thesis (concluded)

- All have been proved to be equivalent.
- Church's thesis is also called the **Church-Turing Thesis**.

Alonso Church (1903–1995)

Extended Church's Thesis[a]

- All "reasonably succinct encodings" of problems are *polynomially related* (e.g., n^2 vs. n^6).
  - Representations of a graph as an adjacency matrix and as a linked list are both succinct.
  - The *unary* representation of numbers is not succinct.
  - The *binary* representation of numbers is succinct.
    * 1001_2 vs. 111111111_1.
- All numbers for TMs will be binary from now on.

---

[a]Some call it "polynomial Church's thesis," which Lószló Lovász attributed to Leonid Levin.

Extended Church's Thesis (concluded)

- Representations that are not succinct may give misleadingly low complexities.
  - Consider an algorithm with binary inputs that runs in 2^n steps.
  - Suppose the input uses unary representation instead.
  - Then the same algorithm runs in linear time because the input length is now 2^n!
- So a succinct representation means honest accounting.

Physical Church-Turing Thesis

- The **physical Church-Turing thesis** states that:

  Anything computable in physics can also be computed on a Turing machine.[a]

- The universe is a Turing machine.[b]

---

[a] Cooper (2012).
[b] Edward Fredkin's (1992) digital physics.

The Strong Church-Turing Thesis[a]

- The **strong Church-Turing thesis** states that:

  A Turing machine can compute *any* function computable by any "reasonable" physical device with only polynomial slowdown.[b]

- A CPU, a GPU, and a DSP chip are good examples of physical devices.[c]

---

[a] Vergis, Steiglitz, & Dickinson (1986).

[b] `http://ocw.mit.edu/courses/mathematics/18-405j-advanced-complexity-theory-fall-2001/lecture-notes/lecture10.pdf`

[c] Thanks to a lively discussion on September 23, 2014.

The Strong Church-Turing Thesis (continued)

- Factoring is believed to be a hard problem for Turing machines (but there is no proof yet).
- But a quantum computer can factor numbers in probabilistic polynomial time.[a]
- If a large-scale stable quantum computer can be reliably built, the strong Church-Turing thesis may be refuted.[b]

[a] Shor (1994).

[b] Contributed by Mr. Kai-Yuan Hou (B99201038, R03922014) on September 22, 2015.

The Strong Church-Turing Thesis (concluded)

- As of 2919,[a]

> There is no publicly known application of commercial interest based upon quantum algorithms that could be run on a near-term analog or digital NISQ[b] computer that would provide an advantage over classical approaches.

---

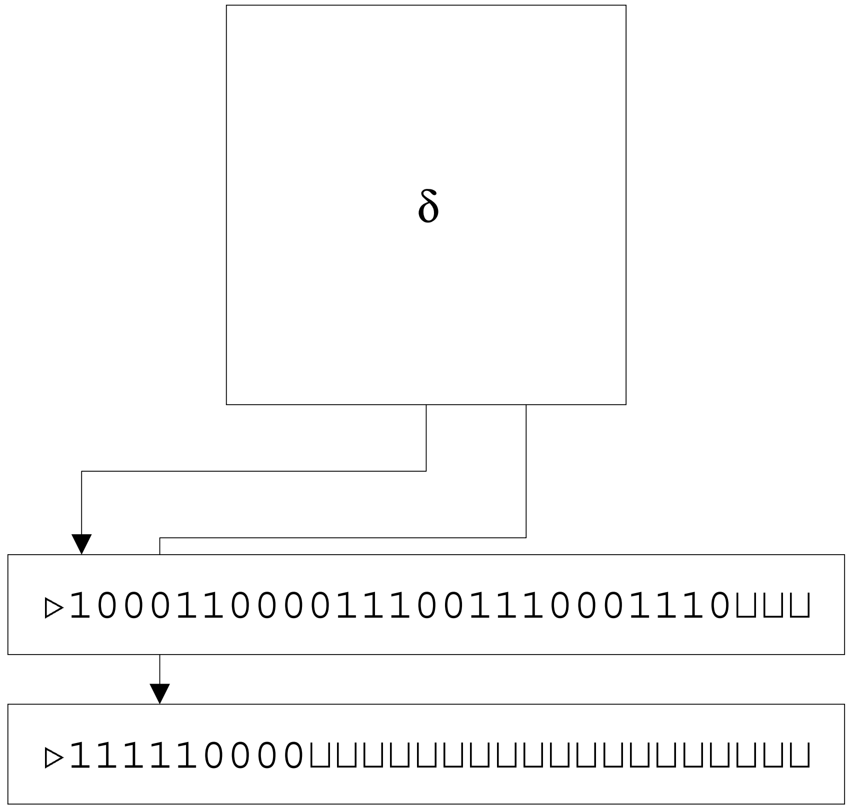
[a] Grumbling & Horowitz (2019).

[b] "Noisy, Intermediate-Scale Quantum."

Turing Machines with Multiple Strings

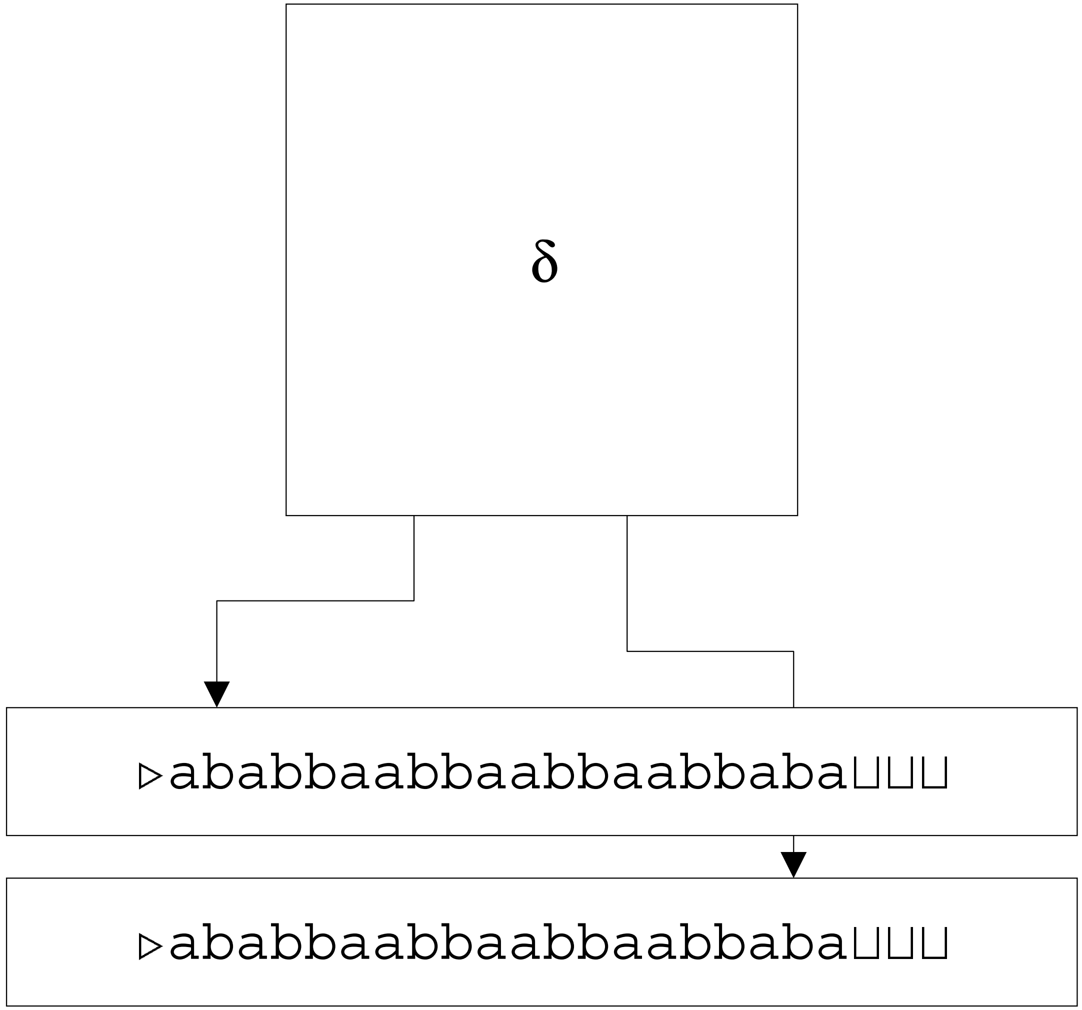
- A k-string Turing machine (TM) is a quadruple $M = (K, \Sigma, \delta, s)$.
- K, Σ, s are as before.
- $\delta : K \times \Sigma^k \to (K \cup \{h, \text{"yes"}, \text{"no"}\}) \times (\Sigma \times \{\leftarrow, \rightarrow, -\})^k$.
- All strings start with a $\triangleright$.
- The first string contains the input.
- Decidability and acceptability are the same as before.
- When TMs compute functions, the output is the last (kth) string.

A 2-String TM



PALINDROME Revisited

- A 2-string TM can decide PALINDROME in $O(n)$ steps.
  - It copies the input to the second string.
  - The cursor of the first string is positioned at the first symbol of the input.
  - The cursor of the second string is positioned at the last symbol of the input.
  - The symbols under the cursors are then compared.
  - The two cursors are then moved in opposite directions until the ends are reached.
  - The machine accepts if and only if the symbols under the two cursors are identical at all steps.


δ
▷ababbaabbaabbaabbaba⊔⊔⊔
▷ababbaabbaabbaabbaba⊔⊔⊔

PALINDROME Revisited (concluded)

- The running times of a 2-string TM and a single-string TM are quadratically related: n^2 vs. n.
- This is consistent with the extended Church's thesis (p. 67).
  - "Reasonable" models are related polynomially in running times.

Configurations and Yielding

- The concept of configuration and yielding is the same as before except that a configuration is a $(2k+1)$-tuple

$$(q, w_1, u_1, w_2, u_2, \ldots, w_k, u_k).$$

  - $w_i u_i$ is the ith string.
  - The ith cursor is reading the last symbol of w_i.
  - Recall that $\triangleright$ is each w_i's first symbol.

- The k-string TM's initial configuration is

$$(s, \overbrace{\underbrace{\triangleright, x}_{1}, \underbrace{\triangleright, \epsilon}_{2}, \underbrace{\triangleright, \epsilon}_{3}, \ldots, \underbrace{\triangleright, \epsilon}_{k}}^{2k}).$$

Time seemed to be
the most obvious measure
of complexity.
— Stephen Arthur Cook (1939–)

Time Complexity

- The multistring TM is the basis of our notion of the time expended by TMs.
- If a k-string TM M halts after t steps on input x, then the **time required by M on input x** is t.
- If $M(x) = \nearrow$, then the time required by M on x is ∞.

Time Complexity (concluded)

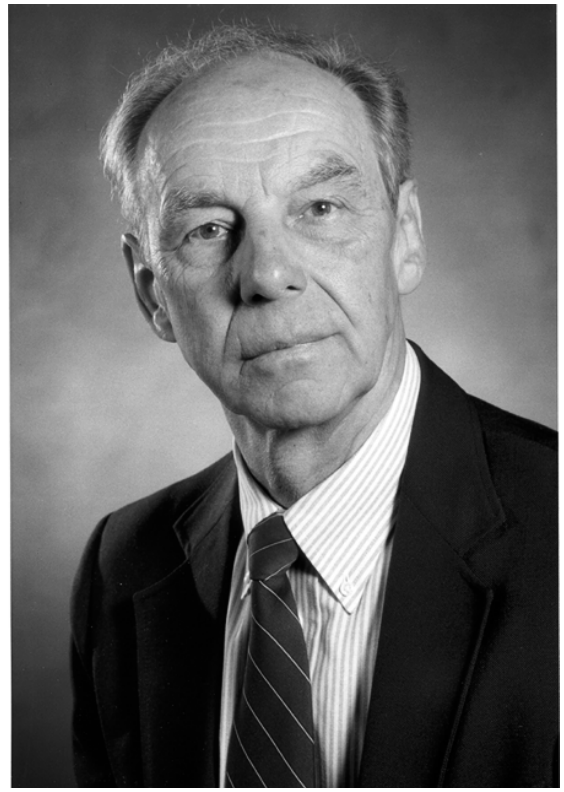
- Machine M **operates within time** $f(n)$ for $f : \mathbb{N} \to \mathbb{N}$ if for *any* input string x, the time required by M on x is at most $f(\,|\,x\,|\,)$.
  - $|\,x\,|$ is the length of string x.
- Function $f(n)$ is a **time bound** for M.

Time Complexity Classes[a]

- Suppose language $L \subseteq (\Sigma - \{\sqcup\})^*$ is decided by a multistring TM operating in time $f(n)$.
- We say $L \in \text{TIME}(f(n))$.
- $\text{TIME}(f(n))$ is the set of languages decided by TMs with multiple strings operating within time bound $f(n)$.
- $\text{TIME}(f(n))$ is a **complexity class**.
  - PALINDROME is in $\text{TIME}(f(n))$, where $f(n) = O(n)$.
- Trivially, $\text{TIME}(f(n)) \subseteq \text{TIME}(g(n))$ if $f(n) \leq g(n)$ for all n.

[a] Hartmanis & Stearns (1965); Hartmanis, Lewis, & Stearns (1965).

Juris Hartmanis[a] (1928–)



[a]Turing Award (1993).

Richard Edwin Stearns[a] (1936–)

[a]Turing Award (1993).

The Simulation Technique

Theorem 3 *Given any k-string M operating within time $f(n)$, there exists a (single-string) M' operating within time $O(f(n)^2)$ such that $M(x) = M'(x)$ for any input x.*

- The single string of M' implements the k strings of M.

The Proof

- Represent configuration $(q, w_1, u_1, w_2, u_2, \ldots, w_k, u_k)$ of M by this string of M':

$$(q, \rhd w_1' u_1 \lhd w_2' u_2 \lhd \cdots \lhd w_k' u_k \lhd \lhd).$$

  - $\lhd$ is a special delimiter.
  - w_i' is w_i with the first[a] and last symbols "primed."
  - It serves the purpose of "," in a configuration.[b]

---

[a] The first symbol is of course $\rhd$.

[b] An alternative is to use $(q, \rhd w_1' | u_1 \lhd w_2' | u_2 \lhd \cdots \lhd w_k' | u_k \lhd \lhd)$ by priming only $\rhd$ in w_i, where "$|$" is a new symbol.

The Proof (continued)

- The first symbol of w_i' is the primed version of $\triangleright$: $\triangleright'$.
  - Recall TM cursors are not allowed to move to the left of $\triangleright$ (p. 23).
  - Now the cursor of M' can move *between* the simulated strings of M.[a]
- The "priming" of the last symbol of each w_i ensures that M' knows which symbol is under each cursor of M.[b]

---

[a] Thanks to a lively discussion on September 22, 2009.

[b] Added because of comments made by Mr. Che-Wei Chang (`R95922093`) on September 27, 2006.

The Proof (continued)

- The initial configuration of M' is

$$(s, \rhd \rhd'' x \lhd \overbrace{\rhd'' \lhd \cdots \rhd'' \lhd}^{k-1 \text{ pairs}} \lhd).$$

  - $\rhd''$ is double-primed because it is the beginning and the ending symbol as the cursor is reading it.[a]
  - Again, think of it as a new symbol.

[a] Added after the class discussion on September 20, 2011.

The Proof (continued)

- We simulate each move of M thus:
  1. M' scans the string to pick up the k symbols under the cursors.
     - The states of M' must be enlarged to include $K \times \Sigma^k$ to remember them.[a]
     - The transition functions of M' must also reflect it.
  2. M' then changes the string to reflect the overwriting of symbols and cursor movements of M.

---

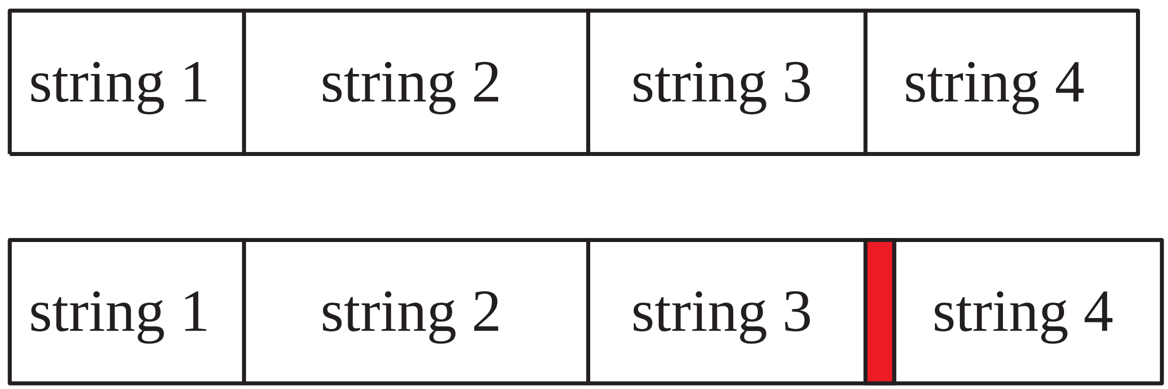
[a] Recall the TM program on p. 32.

The Proof (continued)

- It is possible that some strings of M need to be lengthened (see next page).
  - The linear-time algorithm on p. 38 can be used for each such string.
- The simulation continues until M halts.
- M' then erases all strings of M except the last one.[a]

[a] Because whatever appears on the string of M' will be considered the output. So $\rhd'$s and $\rhd''$s need to be removed.

The Proof (continued)[a]



[a]If we interleave the strings, the simulation may be easier. Contributed by Mr. Kai-Yuan Hou (`B99201038`, `R03922014`) on September 22, 2015. This is similar to constructing a single-string *multi-track* TM in, e.g., Hopcroft & Ullman (1969).

The Proof (continued)

- Since M halts within time $f(|x|)$, none of its strings ever becomes longer than $f(|x|)$.[a]
- The length of the string of M' at any time is $O(kf(|x|))$.
- Simulating each step of M takes, *per string of* M, $O(kf(|x|))$ steps.
  - $O(f(|x|))$ steps to collect information from this string.
  - $O(kf(|x|))$ steps to write and, if needed, to lengthen the string.

[a] We tacitly assume $f(n) \geq n$.

The Proof (concluded)

- M' takes $O(k^2 f(|x|))$ steps to simulate each step of M because there are k strings.
- As there are $f(|x|)$ steps of M to simulate, M' operates within time $O(k^2 f(|x|)^2)$.[a]

[a] Is the time reduced to $O(kf(|x|)^2)$ if the interleaving data structure is adopted?

Simulation with Two-String TMs

We can do better with two-string TMs.

Theorem 4 *Given any k-string M operating within time $f(n)$, $k > 2$, there exists a two-string M' operating within time $O(f(n) \log f(n))$ such that $M(x) = M'(x)$ for any input x.*

Linear Speedup[a]

Theorem 5 *Let $L \in \mathit{TIME}(f(n))$. Then for any $\epsilon > 0$, $L \in \mathit{TIME}(f'(n))$, where $f'(n) \triangleq \epsilon f(n) + n + 2$.*

See Theorem 2.2 of the textbook for a proof.

[a]Hartmanis & Stearns (1965).

Implications of the Speedup Theorem

- State size can be traded for speed.[a]
- If the running time is cn with $c > 1$, then c can be made arbitrarily close to 1.
- If the running time is superlinear, say $14n^2 + 31n$, then the constant in the leading term (14 in this example) can be made arbitrarily small.
  - *Arbitrary* linear speedup can be achieved.[b]
  - This justifies the big-O notation in the analysis of algorithms.

---

[a] $m^k \cdot |\Sigma|^{3mk}$-fold increase to gain a speedup of $O(m)$. No free lunch.

[b] Can you apply the theorem multiple times to achieve superlinear speedup? Thanks to a question by a student on September 21, 2010.

P

- By the linear speedup theorem, any polynomial time bound can be represented by its leading term n^k for some $k \geq 1$.
- If $L \in \text{TIME}(n^k)$ for some $k \in \mathbb{N}$, it is a **polynomially decidable language**.
  - Clearly, $\text{TIME}(n^k) \subseteq \text{TIME}(n^{k+1})$.
- The union of all polynomially decidable languages is denoted by P:

$$\text{P} \stackrel{\Delta}{=} \bigcup_{k>0} \text{TIME}(n^k).$$

- P contains problems that can be efficiently solved.

Philosophers have explained space.
They have not explained time.
— Arnold Bennett (1867–1931),
How To Live on 24 Hours a Day (1910)

I keep bumping into that silly quotation
attributed to me that says
640K of memory is enough.
— Bill Gates (1996)

Space Complexity

- Consider a k-string TM M with input x.
- Assume non-$\sqcup$ is never written over by $\sqcup$.[a]
  - The purpose is not to artificially reduce the space needs (see below).
- If M halts in configuration

$$(H, w_1, u_1, w_2, u_2, \ldots, w_k, u_k),$$

then the **space required by M on input x** is

$$\sum_{i=1}^{k} |\, w_i u_i \,|.$$

[a]Corrected by Ms. Chuan-Ju Wang (R95922018, F95922018) on September 27, 2006.

Space Complexity (continued)

- Suppose we do not charge the space used only for input and output.
- Let $k > 2$ be an integer.
- A **k-string Turing machine with input and output** is a k-string TM that satisfies the following conditions.
  - The input string is *read-only*.[a]
  - The cursor on the last string never moves to the left.
    * The output string is essentially *write-only*.
  - The cursor of the input string does not wander off into the $\sqcup$s.

---

[a] Called an **off-line** TM in Hartmanis, Lewis, & Stearns (1965).

Space Complexity (concluded)

- If M is a TM with input and output, then the space required by M on input x is

$$\sum_{i=2}^{k-1} |\, w_i u_i \,|.$$

- Machine M **operates within space bound** $f(n)$ for $f : \mathbb{N} \to \mathbb{N}$ if for any input x, the space required by M on x is at most $f(|\, x \,|)$.

Space Complexity Classes

- Let L be a language.
- Then
$$L \in \text{SPACE}(f(n))$$
if there is a TM with input and output that decides L and operates within space bound $f(n)$.
- SPACE$(f(n))$ is a set of languages.
  - PALINDROME $\in$ SPACE$(\log n)$.[a]
- A linear speedup theorem similar to the one on p. 95 exists, so constant coefficients do not matter.

---
[a]Keep 3 counters.

If she can hesitate as to "Yes,"
she ought to say "No" directly.
— Jane Austen (1775–1817),
Emma (1815)

Nondeterminism[a]

- A **nondeterministic Turing machine** (**NTM**) is a quadruple $N = (K, \Sigma, \Delta, s)$.
- K, Σ, s are as before.
- $\Delta \subseteq K \times \Sigma \times (K \cup \{h, \text{"yes"}, \text{"no"}\}) \times \Sigma \times \{\leftarrow, \rightarrow, -\}$ is a *relation*, not a *function*.[b]
  - For each state-symbol combination (q, σ), there may be *multiple* valid next steps.
  - Multiple lines of code may be applicable.
  - But only one will be taken.

[a] Rabin & Scott (1959).

[b] Corrected by Mr. Jung-Ying Chen (D95723006) on September 23, 2008.

Nondeterminism (continued)

- As before, a program contains lines of code:

$$
\begin{aligned}
(q_1, \sigma_1, p_1, \rho_1, D_1) &\in \Delta, \\
(q_2, \sigma_2, p_2, \rho_2, D_2) &\in \Delta, \\
&\vdots \\
(q_n, \sigma_n, p_n, \rho_n, D_n) &\in \Delta.
\end{aligned}
$$

- But we cannot write

$$\delta(q_i, \sigma_i) = (p_i, \rho_i, D_i)$$

as in the deterministic case (p. 24) anymore.

Nondeterminism (concluded)

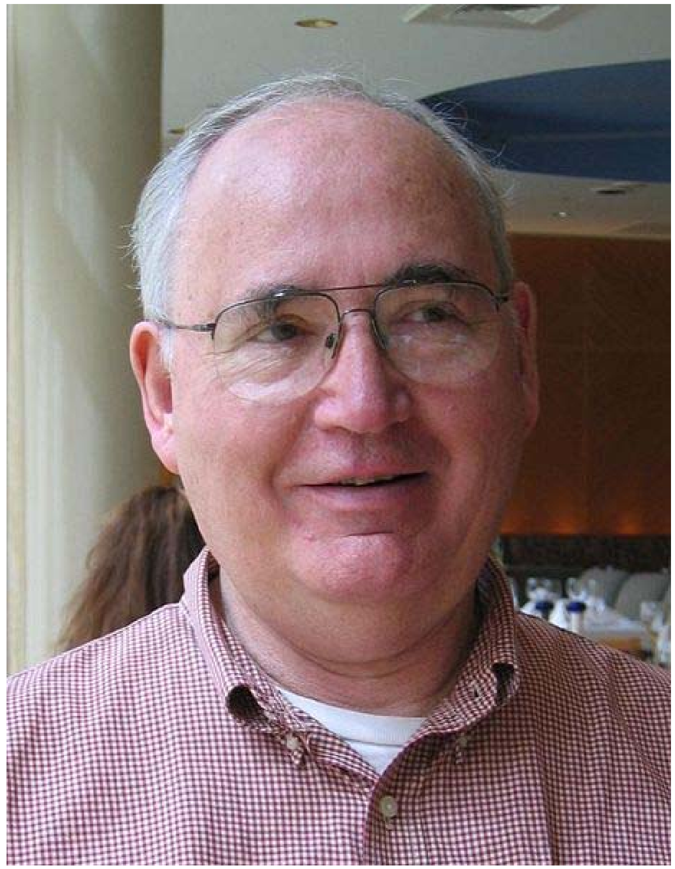
- A configuration yields another configuration in one step if there *exists* a rule in Δ that makes this happen.
- There is only a single thread of computation.[a]
  - Nondeterminism is *not* parallelism, multiprocessing, multithreading, or quantum computation.

[a] Thanks to a lively discussion on September 22, 2015.

Michael O. Rabin[a] (1931–)

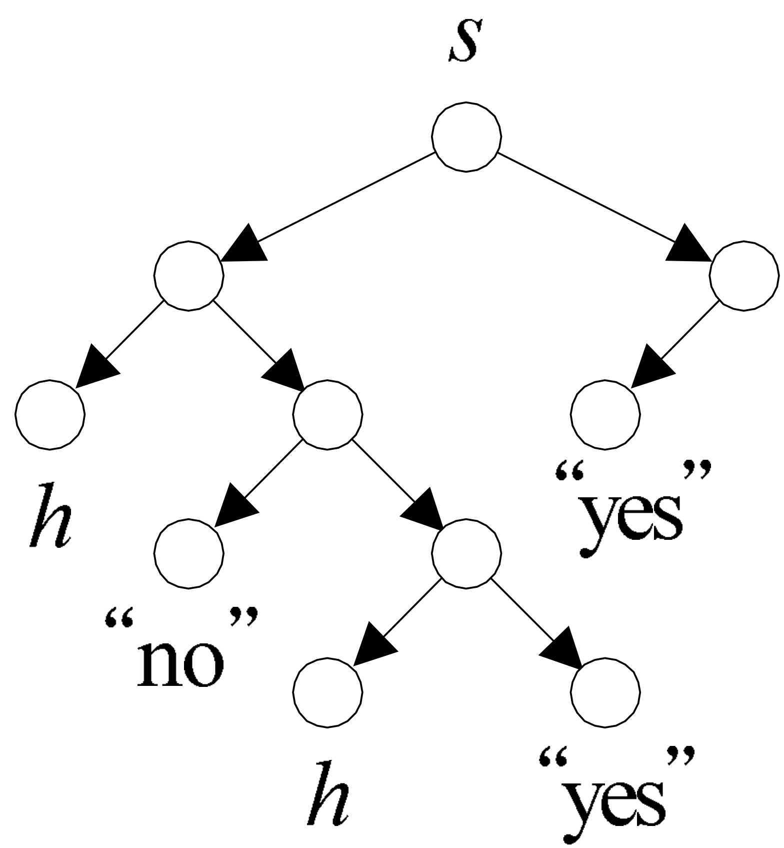
[a]Turing Award (1976).

Dana Stewart Scott[a] (1932–)



[a]Turing Award (1976).

Computation Tree and Computation Path



Decidability under Nondeterminism

- Let L be a language and N be an NTM.
- N **decides** L if for any $x \in \Sigma^*$, $x \in L$ if and only if there is a sequence of valid configurations that ends in "yes."
- In other words,
  - If $x \in L$, then $N(x) =$ "yes" for some computation path.
  - If $x \notin L$, then $N(x) \neq$ "yes" for all computation paths.

Decidability under Nondeterminism (continued)

- It is not required that the NTM halts in all computation paths.[a]
- If $x \notin L$, no nondeterministic choices should lead to a "yes" state.
- The key is the algorithm's *overall* behavior not whether it gives a correct answer for each particular run.
- Note that determinism is a special case of nondeterminism.

[a]So "accepts" may be a more proper term. Some books use "decides" only when the NTM always halts.

Decidability under Nondeterminism (concluded)

- For example, suppose L is the set of primes.[a]
- Then we have the primality testing problem.
- An NTM N decides L if:
  - If x is a prime, then $N(x) =$ "yes" for some computation path.
  - If x is not a prime, then $N(x) \neq$ "yes" for all computation paths.

[a]Contributed by Mr. Yu-Ming Lu (R06723032, D08922008) on March 7, 2019.