Computing Theory

Midterm Examination on June 20, 2019
Spring Semester, 2019

Problem 1 (25 points) The class **BPP** contains all languages L for which there is a precise polynomial-time NTM N satisfying prob$[\, N(x) = L(x) \,] \geq 3/4$ for all x. (That is, if $x \in L$, then at least 3/4 of the computation paths of N on x lead to "yes." Otherwise, at least 3/4 of the computation paths of N on x lead to "no.") Let p-**BPP** denote the class of languages L for which there is a precise polynomial-time NTM N satisfying prob$[\, N(x) = L(x) \,] \geq 1/2 + 1/p(n)$ for all x, where $p(n)$ is a polynomial function and $n = |x|$. Prove that **BPP** $=$ p-**BPP**. (Hint: Recall that one version of the Chernoff bound is that prob$[\, \sum_{i=1}^{n} x_i \leq n/2 \,] \leq e^{-\epsilon^2 n/2}$ where $p = 1/2 + \epsilon$ for any $0 \leq \epsilon \leq 1/2$.)

Proof: It is obvious that **BPP** $\subseteq$ p-**BPP**. We will show that p-**BPP** $\subseteq$ **BPP**. For all languages L in p-**BPP**, for every input $x \in L$, run machine N for k times. The Chernoff bound $\Pr[\sum_{i=1}^{k} x_i \leq k/2] \leq e^{-\epsilon^2 k/2}$ implies that the probability of a false answer is at most $e^{-\epsilon^2 k/2}$. By taking $k = \lceil 2p(n)^2 \ln 4 \rceil$, the error probability is at most 1/4. ■

Problem 2 (25 points) Consider the interactive proof system for GRAPH NONISOMORPHISM in the lecture. Suppose Bob's behavior is the same except that he flips his answer in the end: he accepts if and only if the original protocol rejects. Is the resulting system an interactive proof system for GRAPH ISOMORPHISM? Why?

Proof: To be a system for GRAPH ISOMORPHISM, it must satisfy (1) if G, G' are isomorphic then accepted with extremely high probability, and (2) if they are not isomorphic, then reject with extremely high probability regardless of the prover. For (1), suppose G, G' are isomorphic. Then Alice sees all graphs as the same, and she answers one always. Hence all the answers will be 1s. For (2), suppose G, G' are not isomorphic. Then does Bob's strategy prevent cheating? No, because Alice can cheat Bob by always answering 1s only! Hence, it is *not* a system for GRAPH ISOMORPHISM. ■

Problem 3 (25 points) Given three disjoint sets A, B, and C, each containing n elements, and a ternary relation $T \subseteq A \times B \times C$, a tripartite matching is a set of triples in T, none of which has an element in common. The problem MAXIMUM TRIPARTITE MATCHING seeks the largest tripartite matching. There is an approximation algorithm for MAXIMUM TRIPARTITE MATCHING:

1: $M := \emptyset$
2: **while** there is a triple (a, b, c) in T such that $a \in A, b \in B, c \in C$ **do**
3: Add (a, b, c) to M;

4: Delete a, b, c from A, B, C, respectively;
5: **end while**
6: **return** M;

Prove that the above algorithm is a 2/3-approximation algorithm. (That is, prove that the approximation ratio of the above algorithm is $c(M(x))/\text{OPT}(x) \geq 1/3$.)

Proof: Let M^* be a maximum tripartite matching. Since we can not add any triple from M to M^* when the algorithm ends, each triple in M^* must have at least one element in common with some element in M. Because all triples in M^* contain disjoint elements, the size of M^* is upper-bounded by the number of elements of all the triples in M. That is, $|M^*| \leq 3|M|$. The approximation ratio is hence $|M|/|M^*| \geq 1/3$. ■

Problem 4 (25 points) Prove that a k-SAT expression where literals in a clause (which contains k literals by definition) are distinct must be satisfiable if it has fewer than 2^k clauses. (Hint: Consider a random truth assignment that independently assigns `true` to every variable with probability 1/2. Calculate its expected number of unsatisfiable clauses.)

Proof: Consider a random truth assignment that independently assigns `true` to every variable with probability 1/2. Let X count the number of *unsatisfiable* clauses. It suffices to prove $E[X] < 1$: $\text{prob}[X = 0] = 0$ implies $E[X] \geq 1$ because X is integer-valued. Let

$$\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$$

be such a k-SAT expression with $m < 2^k$ clauses. Define

$$X_i = \begin{cases} 1, & \text{if } C_i \text{ is unsatisfied,} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly

$$X = \sum_{i=1}^{m} X_i.$$

Clearly,

$$E[X_i] \leq 2^{-k}$$

as X_i is a disjunction of distinct literals.[a] Hence

$$E[X] = \sum_{i=1}^{m} E[X_i] \leq m2^{-k} < 1,$$

as desired. ■

[a] If x and $\bar{x}$ both appear in a literal, then the probability that C_i is unsatisfiable is zero, hence the inequality.