Large Deviations

- Suppose you have a *biased* coin.
- One side has probability $0.5 + \epsilon$ to appear and the other $0.5 - \epsilon$, for some $0 < \epsilon < 0.5$.
- But you do not know which is which.
- How to decide which side is the more likely side—with high confidence?
- Answer: Flip the coin many times and pick the side that appeared the most times.
- Question: Can you quantify your confidence?

The (Improved) Chernoff Bound[a]

Theorem 75 (Chernoff, 1952) *Suppose* $x_1, x_2, \ldots, x_n$ *are independent random variables taking the values 1 and 0 with probabilities* p *and* $1-p$*, respectively. Let* $X = \sum_{i=1}^{n} x_i$*. Then for all* $0 \leq \theta \leq 1$*,*

$$\text{prob}[\, X \geq (1+\theta)\, pn \,] \leq e^{-\theta^2 pn/3}.$$

- The probability that the deviate of a **binomial random variable** from its expected value $E[\, X \,] = E\left[\, \sum_{i=1}^{n} x_i \,\right] = pn$ decreases exponentially with the deviation.

[a]Herman Chernoff (1923–). This bound is asymptotically optimal. The original bound is $e^{-2\theta^2 p^2 n}$ (McDiarmid, 1998).

The Proof

- Let t be any positive real number.
- Then

$$\text{prob}[\, X \geq (1+\theta)\, pn \,] = \text{prob}[\, e^{tX} \geq e^{t(1+\theta)\, pn} \,].$$

- Markov's inequality (p. 536) generalized to real-valued random variables says that

$$\text{prob}\left[\, e^{tX} \geq kE[\, e^{tX} \,] \,\right] \leq 1/k.$$

- With $k = e^{t(1+\theta)\, pn} / E[\, e^{tX} \,]$, we have[a]

$$\text{prob}[\, X \geq (1+\theta)\, pn \,] \leq e^{-t(1+\theta)\, pn} E[\, e^{tX} \,].$$

---

[a]Note that X does not appear in k. Contributed by Mr. Ao Sun (`R05922147`) on December 20, 2016.

The Proof (continued)

- Because $X = \sum_{i=1}^n x_i$ and x_i's are independent,

$$E[\, e^{tX}\,] = (E[\, e^{tx_1}\,])^n = [\, 1 + p(e^t - 1)\,]^n.$$

- Substituting, we obtain

$$\begin{aligned} \text{prob}[\, X \geq (1+\theta)\, pn\,] &\leq e^{-t(1+\theta)\, pn} [\, 1 + p(e^t - 1)\,]^n \\ &\leq e^{-t(1+\theta)\, pn} e^{pn(e^t - 1)} \end{aligned}$$

as $(1 + a)^n \leq e^{an}$ for all $a > 0$.

The Proof (concluded)

- With the choice of $t = \ln(1+\theta)$, the above becomes

$$\text{prob}[\, X \geq (1+\theta)\, pn\,] \leq e^{pn[\,\theta-(1+\theta)\ln(1+\theta)\,]}.$$

- The exponent expands to[a]

$$-\frac{\theta^2}{2}+\frac{\theta^3}{6}-\frac{\theta^4}{12}+\cdots$$

for $0 \leq \theta \leq 1$.

- But it is less than

$$-\frac{\theta^2}{2}+\frac{\theta^3}{6} \leq \theta^2\left(-\frac{1}{2}+\frac{\theta}{6}\right) \leq \theta^2\left(-\frac{1}{2}+\frac{1}{6}\right) = -\frac{\theta^2}{3}.$$

---

[a]Or McDiarmid (1998): $x-(1+x)\ln(1+x) \leq -3x^2/(6+2x)$ for all $x \geq 0$.

Other Variations of the Chernoff Bound

The following can be proved similarly (prove it).

Theorem 76 *Given the same terms as Theorem 75 (p. 599),*

$$\text{prob}[\, X \leq (1-\theta)\, pn \,] \leq e^{-\theta^2 pn/2}.$$

The following slightly looser inequalities achieve symmetry.

Theorem 77 (Karp, Luby, & Madras, 1989) *Given the same terms as Theorem 75 (p. 599) except with* $0 \leq \theta \leq 2$,

$$\begin{aligned} \text{prob}[\, X \geq (1+\theta)\, pn \,] &\leq e^{-\theta^2 pn/4}, \\ \text{prob}[\, X \leq (1-\theta)\, pn \,] &\leq e^{-\theta^2 pn/4}. \end{aligned}$$

Power of the Majority Rule

The next result follows from Theorem 76 (p. 603).

Corollary 78 *If* $p = (1/2) + \epsilon$ *for some* $0 \leq \epsilon \leq 1/2$*, then*

$$\text{prob}\left[\sum_{i=1}^{n} x_i \leq n/2\right] \leq e^{-\epsilon^2 n/2}.$$

- The textbook's corollary to Lemma 11.9 seems too loose, at $e^{-\epsilon^2 n/6}$.[a]
- Our original problem (p. 598) hence demands, e.g., $n \approx 1.4k/\epsilon^2$ independent coin flips to guarantee making an error with probability $\leq 2^{-k}$ with the majority rule.

[a]See Dubhashi & Panconesi (2012) for many Chernoff-type bounds.

BPP[a] (Bounded Probabilistic Polynomial)

- The class **BPP** contains all languages L for which there is a precise polynomial-time NTM N such that:
  - If $x \in L$, then at least $3/4$ of the computation paths of N on x lead to "yes."
  - If $x \notin L$, then at least $3/4$ of the computation paths of N on x lead to "no."
- So N accepts or rejects by a *clear* majority.

[a]Gill (1977).

Magic 3/4?

- The number 3/4 bounds the probability (ratio) of a right answer away from 1/2.
- Any constant *strictly* between 1/2 and 1 can be used without affecting the class BPP.
- In fact, as with RP,

$$\frac{1}{2} + \frac{1}{q(n)}$$

  for any polynomial $q(n)$ can replace 3/4.
- The next algorithm shows why.

The Majority Vote Algorithm

Suppose L is decided by N by majority $(1/2) + \epsilon$.

```
for i = 1, 2, ..., 2k + 1 do
    Run N on input x;
end for
if "yes" is the majority answer then
    "yes";
else
    "no";
end if
```

Analysis

- By Corollary 78 (p. 604), the probability of a false answer is at most $e^{-\epsilon^2 k}$.
- By taking $k = \lceil 2/\epsilon^2 \rceil$, the error probability is at most $1/4$.
- Even if ϵ is any inverse polynomial, k remains a polynomial in n.
- The running time remains polynomial: $2k + 1$ times N's running time.

Aspects of BPP

- BPP is the most comprehensive yet plausible notion of efficient computation.
  - If a problem is in BPP, we take it to mean that the problem can be solved efficiently.
  - In this aspect, BPP has effectively replaced P.
- $(\text{RP} \cup \text{coRP}) \subseteq (\text{NP} \cup \text{coNP})$.
- $(\text{RP} \cup \text{coRP}) \subseteq \text{BPP}$.
- Whether $\text{BPP} \subseteq (\text{NP} \cup \text{coNP})$ is unknown.
- But it is unlikely that $\text{NP} \subseteq \text{BPP}$.[a]

[a]See p. 621.

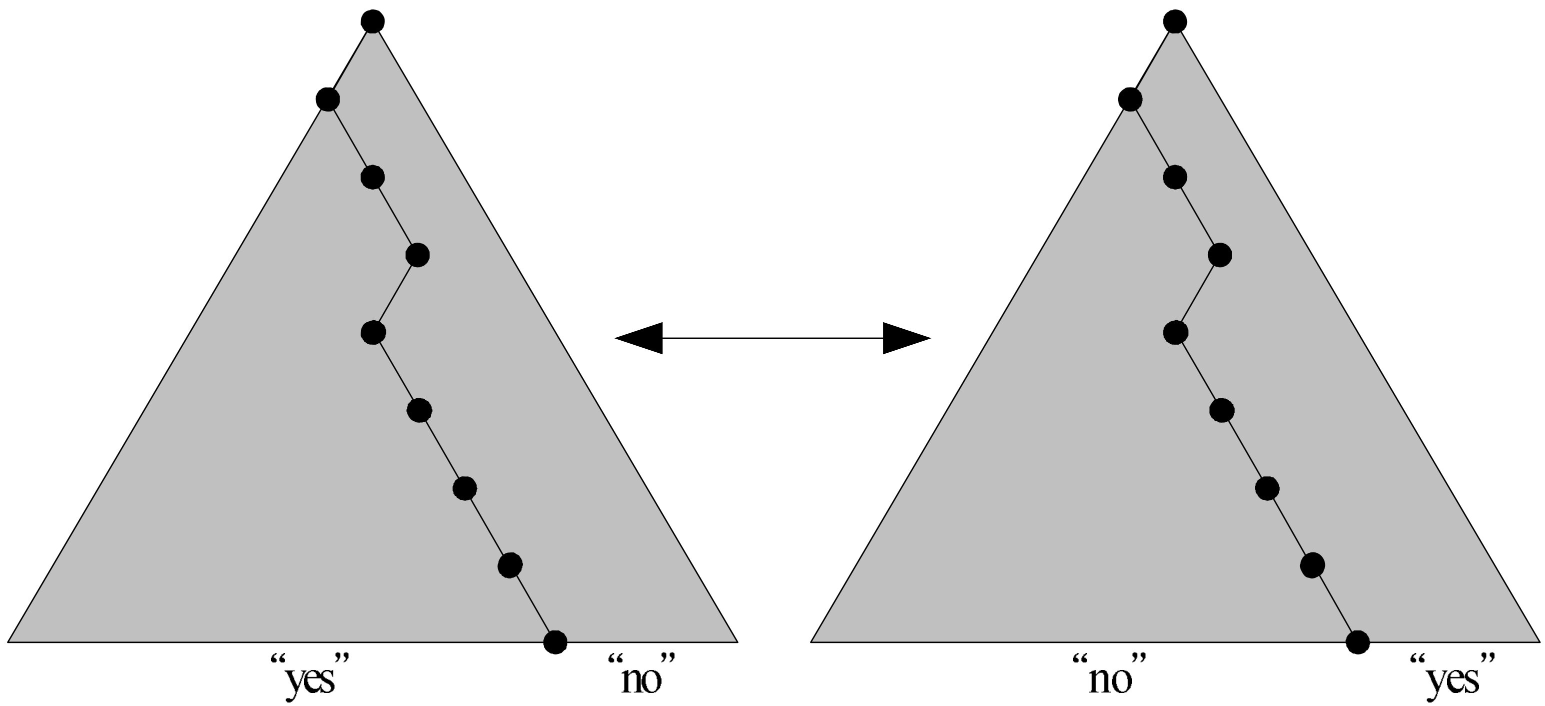
coBPP

- The definition of BPP is symmetric: acceptance by clear majority and rejection by clear majority.
- An algorithm for $L \in$ BPP becomes one for $\bar{L}$ by reversing the answer.
- So $\bar{L} \in$ BPP and BPP $\subseteq$ coBPP.
- Similarly coBPP $\subseteq$ BPP.
- Hence BPP $=$ coBPP.
- This approach does not work for RP.[a]

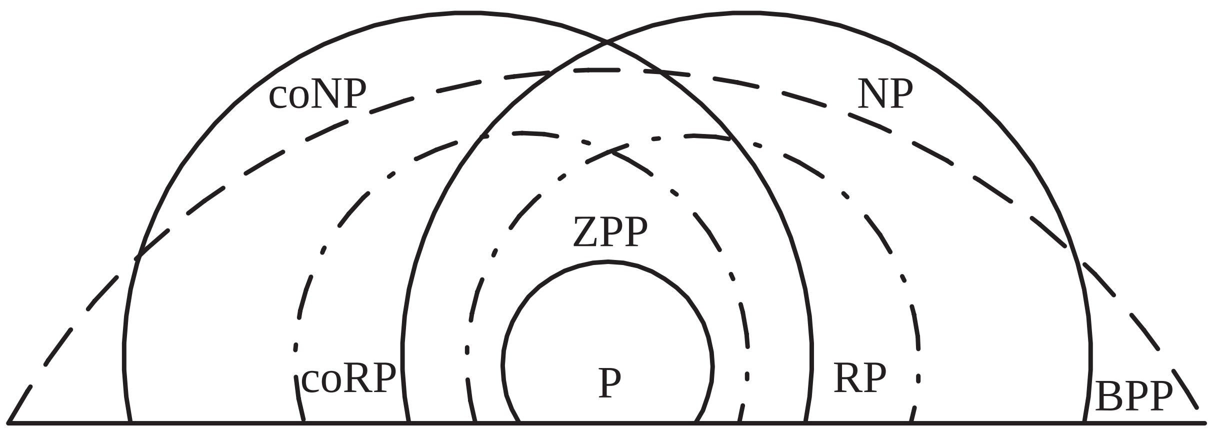
---

[a] It did not work for NP either.

BPP and coBPP



"The Good, the Bad, and the Ugly"



Circuit Complexity

- Circuit complexity is based on boolean circuits instead of Turing machines.
- A boolean circuit with n inputs computes a boolean function of n variables.
- Now, identify `true`/1 with "yes" and `false`/0 with "no."
- Then a boolean circuit with n inputs accepts certain strings in $\{0, 1\}^n$.
- To relate circuits with an arbitrary language, we need one circuit for each possible input length n.

Formal Definitions

- The **size** of a circuit is the number of *gates* in it.
- A **family of circuits** is an infinite sequence $\mathcal{C} = (C_0, C_1, \ldots)$ of boolean circuits, where C_n has n boolean inputs.
- For input $x \in \{0, 1\}^*$, $C_{|x|}$ outputs 1 if and only if $x \in L$.
- In other words,

$$C_n \text{ accepts } L \cap \{0, 1\}^n.$$

Formal Definitions (concluded)

- $L \subseteq \{0,1\}^*$ has **polynomial circuits** if there is a family of circuits $\mathcal{C}$ such that:
  - The size of C_n is at most $p(n)$ for some fixed polynomial p.
  - C_n accepts $L \cap \{0,1\}^n$.

Exponential Circuits Suffice for All Languages

- Theorem 16 (p. 209) implies that there are languages that cannot be solved by circuits of size $2^n/(2n)$.
- But surprisingly, circuits of size 2^{n+2} can solve *all* problems, decidable or otherwise!

Exponential Circuits Suffice for All Languages (continued)

Proposition 79 *All decision problems (decidable or otherwise) can be solved by a circuit of size* 2^{n+2}.

- We will show that for any language $L \subseteq \{0,1\}^*$, $L \cap \{0,1\}^n$ can be decided by a circuit of size 2^{n+2}.
- Define boolean function $f : \{0,1\}^n \to \{0,1\}$, where

$$f(x_1 x_2 \cdots x_n) = \begin{cases} 1, & x_1 x_2 \cdots x_n \in L, \\ 0, & x_1 x_2 \cdots x_n \notin L. \end{cases}$$

The Proof (concluded)

- Clearly, any circuit that implements f decides $L \cap \{0, 1\}^n$.
- Now,

$$f(x_1 x_2 \cdots x_n) = (x_1 \wedge f(1 x_2 \cdots x_n)) \vee (\neg x_1 \wedge f(0 x_2 \cdots x_n)).$$

- The circuit size $s(n)$ for $f(x_1 x_2 \cdots x_n)$ hence satisfies

$$s(n) = 4 + 2s(n-1)$$

  with $s(1) = 1$.
- Solve it to obtain $s(n) = 5 \times 2^{n-1} - 4 \leq 2^{n+2}$.

The Circuit Complexity of P

Proposition 80 *All languages in P have polynomial circuits.*

- Let $L \in \mathrm{P}$ be decided by a TM in time $p(n)$.
- By Corollary 35 (p. 315), there is a circuit with $O(p(n)^2)$ gates that accepts $L \cap \{0, 1\}^n$.
- The size of that circuit depends only on L and the length of the input.
- The size of that circuit is polynomial in n.

Polynomial Circuits vs. P

- Is the converse of Proposition 80 true?
  - Do polynomial circuits accept only languages in P?
- No.
- Polynomial circuits can accept *undecidable* languages![a]

[a] See p. 268 of the textbook.

BPP's Circuit Complexity: Adleman's Theorem

Theorem 81 (Adleman, 1978) *All languages in BPP have polynomial circuits.*

- Our proof will be *nonconstructive* in that only the existence of the desired circuits is shown.
  - Recall our proof of Theorem 16 (p. 209).
  - Something exists if its probability of existence is nonzero.
- It is not known how to efficiently generate circuit C_n.
  - If the construction of C_n can be made efficient, then P = BPP, an unlikely result.

The Proof

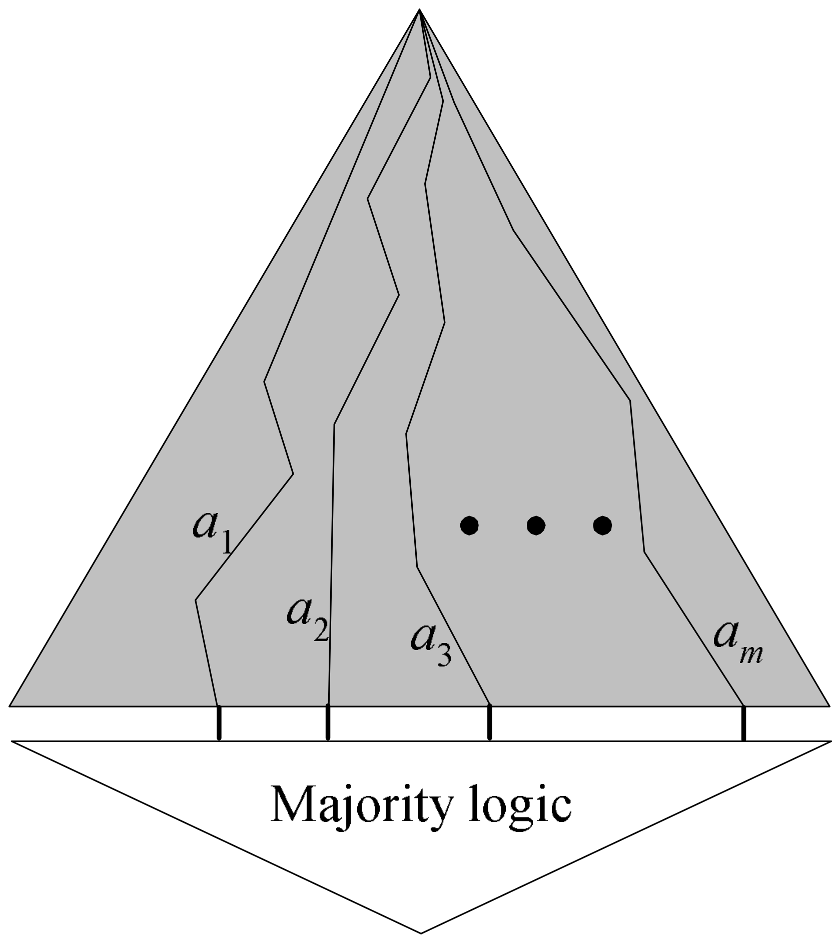
- Let $L \in$ BPP be decided by a precise polynomial-time NTM N by clear majority.
- We shall prove that L has polynomial circuits $C_0, C_1, \ldots$.
  - These *deterministic* circuits do not err.
- Suppose N runs in time $p(n)$, where $p(n)$ is a polynomial.
- Let $A_n = \{ a_1, a_2, \ldots, a_m \}$, where $a_i \in \{ 0, 1 \}^{p(n)}$.
- Each $a_i \in A_n$ represents a sequence of nondeterministic choices (i.e., a computation path) for N.
- Pick $m = 12(n + 1)$.

The Proof (continued)

- Let x be an input with $|\,x\,| = n$.
- Circuit C_n simulates N on x with all sequences of choices in A_n and then takes the majority of the m outcomes.[a]
  - Note that each A_n yields a circuit.
- As N with a_i is a polynomial-time deterministic TM, it can be simulated by polynomial circuits of size $O(p(n)^2)$.
  - See the proof of Proposition 80 (p. 619).

[a]As m is even, there may be no clear majority. Still, the probability of that happening is very small and does not materially affect our general conclusion. Thanks to a lively class discussion on December 14, 2010.

The Circuit



The Proof (continued)

- The size of C_n is therefore $O(mp(n)^2) = O(np(n)^2)$.
  - This is a polynomial.
- We now confirm the existence of an A_n making C_n correct on *all* n-bit inputs.
- Call a_i **bad** if it leads N to an error (a false positive or a false negative) for x.
- Select A_n uniformly randomly.

The Proof (continued)

- For each $x \in \{0, 1\}^n$, $1/4$ of the computations of N are erroneous.
- Because the sequences in A_n are chosen randomly and independently, the expected number of bad a_i's is $m/4$.[a]
- Also note after fixing the input x, the circuit is a function of the random bits.

[a]So the proof will not work for NP. Contributed by Mr. Ching-Hua Yu (`D00921025`) on December 11, 2012.

The Proof (continued)

- By the Chernoff bound (p. 599), the probability that the number of bad a_i's is $m/2$ or more is at most

$$e^{-m/12} < 2^{-(n+1)}.$$

- The error probability of using the majority rule is thus

$$< 2^{-(n+1)}$$

for each $x \in \{0, 1\}^n$.

The Proof (continued)

- The probability that there is an x such that A_n results in an incorrect answer is

$$< 2^n 2^{-(n+1)} = 2^{-1}.$$

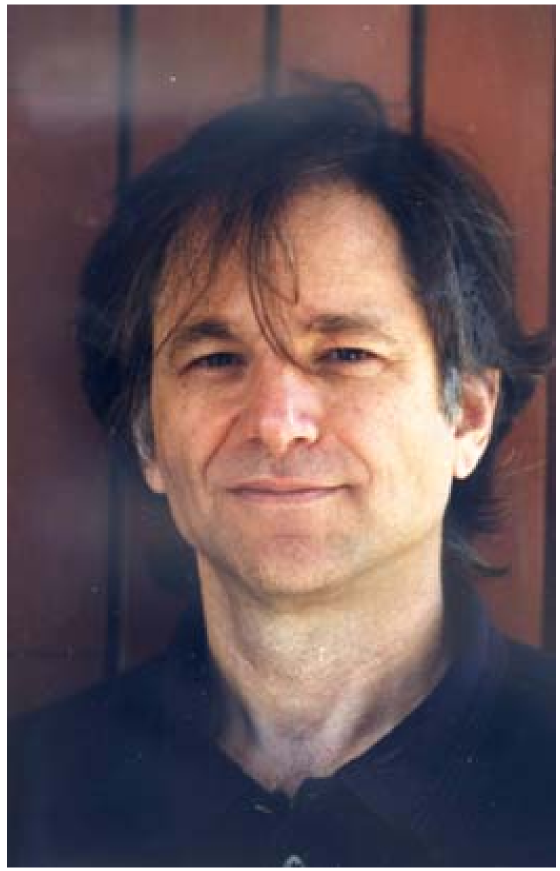
  - Recall the union bound (**Boole's inequality**): $\text{prob}[\, A \cup B \cup \cdots] \leq \text{prob}[\, A\,] + \text{prob}[\, B\,] + \cdots$.
- We just showed that at least half of them are correct.
- So with probability ≥ 0.5, a random A_n produces a correct C_n for *all* inputs of length n.
  - Of course, verifying this fact may take a long time.

The Proof (concluded)

- Because this probability exceeds 0, an A_n that makes majority vote work for *all* inputs of length n exists.
- Hence a correct C_n exists.[a]
- We have used the **probabilistic method**[b] popularized by Erdős (1947).[c]
- This result answers the question on p. 531 with a "yes."

[a]Quine (1948), "To be is to be the value of a bound variable."
[b]A counting argument in the probabilistic language.
[c]Szele (1943) and Turán (1934) were earlier.

Leonard Adleman[a] (1945–)



[a]Turing Award (2002).

Paul Erdős (1913–1996)

Cryptography

Whoever wishes to keep a secret
must hide the fact that he possesses one.
— Johann Wolfgang von Goethe (1749–1832)

Cryptography

- **Alice** (A) wants to send a message to **Bob** (B) over a channel monitored by **Eve** (eavesdropper).
- The protocol should be such that the message is known only to Alice and Bob.
- The art and science of keeping messages secure is **cryptography**.

$$\text{Alice} \xrightarrow{\text{Eve}} \text{Bob}$$

Encryption and Decryption

- Alice and Bob agree on two algorithms E and D—the **encryption** and the **decryption algorithms**.
- Both E and D are known to the public in the analysis.
- Alice runs E and wants to send a message x to Bob.
- Bob operates D.

Encryption and Decryption (concluded)

- Privacy is assured in terms of two numbers e, d, the **encryption** and **decryption keys**.
- Alice sends $y = E(e, x)$ to Bob, who then performs $D(d, y) = x$ to recover x.
- x is called **plaintext**, and y is called **ciphertext**.[a]

[a] Both "zero" and "cipher" come from the same Arab word.

Some Requirements

- D should be an inverse of E given e and d.
- D and E must both run in (probabilistic) polynomial time.
- Eve should not be able to recover x from y without knowing d.
  - As D is public, d must be kept secret.
  - e may or may not be a secret.

Degree of Security

- **Perfect secrecy**: After a ciphertext is intercepted by the enemy, the a posteriori probabilities of the plaintext that this ciphertext represents are identical to the a priori probabilities of the same plaintext before the interception.
  - The probability that plaintext $\mathcal{P}$ occurs is independent of the ciphertext $\mathcal{C}$ being observed.
  - So knowing $\mathcal{C}$ yields no advantage in recovering $\mathcal{P}$.

Degree of Security (concluded)

- Such systems are said to be **informationally secure**.
- A system is **computationally secure** if breaking it is theoretically possible but computationally infeasible.

Conditions for Perfect Secrecy[a]

- Consider a cryptosystem where:
  - The space of ciphertext is as large as that of keys.
  - Every plaintext has a nonzero probability of being used.
- It is **perfectly secure** if and only if the following hold.
  - A key is chosen with uniform distribution.
  - For each plaintext x and ciphertext y, there exists a unique key e such that $E(e, x) = y$.

---

[a] Shannon (1949).

The One-Time Pad[a]

1: Alice generates a random string r as long as x;

2: Alice sends r to Bob over a secret channel;

3: Alice sends $x \oplus r$ to Bob over a public channel;

4: Bob receives y;

5: Bob recovers $x := y \oplus r$;

---

[a]Mauborgne & Vernam (1917); Shannon (1949). It was allegedly used for the hotline between Russia and U.S.

Analysis

- The one-time pad uses $e = d = r$.
- This is said to be a **private-key cryptosystem**.
- Knowing x and knowing r are equivalent.
- Because r is random and private, the one-time pad achieves perfect secrecy.[a]
- The random bit string must be new for each round of communication.
- But the assumption of a private channel is problematic.

[a] See p. 640.

Public-Key Cryptography[a]

- Suppose only d is private to Bob, whereas e is public knowledge.
- Bob generates the (e, d) pair and publishes e.
- Anybody like Alice can send $E(e, x)$ to Bob.
- Knowing d, Bob can recover x via

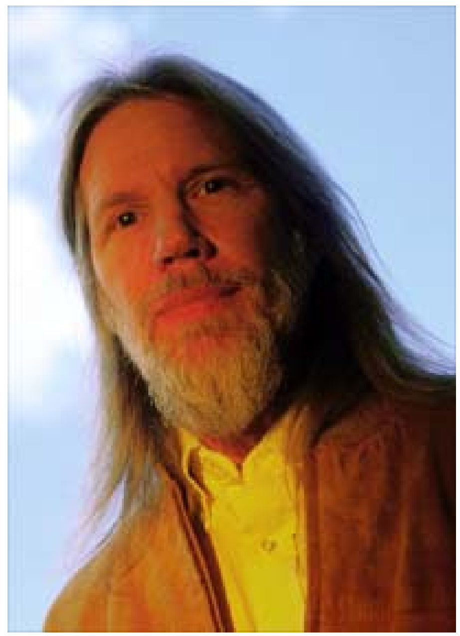
$$D(d, E(e, x)) = x.$$

[a]Diffie & Hellman (1976).

Public-Key Cryptography (concluded)

- The assumptions are complexity-theoretic.
  - It is computationally difficult to compute d from e.
  - It is computationally difficult to compute x from y without knowing d.

Whitfield Diffie[a] (1944–)



[a]Turing Award (2016).

Martin Hellman[a] (1945–)

[a]Turing Award (2016).

Complexity Issues

- Given y and x, it is easy to verify whether $E(e, x) = y$.
- Hence one can always guess an x and verify.
- Cracking a public-key cryptosystem is thus in NP.
- A *necessary* condition for the existence of secure public-key cryptosystems is P $\neq$ NP.
- But more is needed than P $\neq$ NP.
- For instance, it is not sufficient that D is hard to compute in the *worst* case.
- It should be hard in "most" or "average" cases.

One-Way Functions

A function f is a **one-way function** if the following hold.[a]

1. f is one-to-one.
2. For all $x \in \Sigma^*$, $|\,x\,|^{1/k} \leq |f(x)| \leq |\,x\,|^k$ for some $k > 0$.
   - f is said to be **honest**.
3. f can be computed in polynomial time.
4. f^{-1} cannot be computed in polynomial time.
   - Exhaustive search works, but it must be slow.

---

[a] Diffie & Hellman (1976); Boppana & Lagarias (1986); Grollmann & Selman (1988); Ko (1985); Ko, Long, & Du (1986); Watanabe (1985); Young (1983).

Existence of One-Way Functions (OWFs)

- Even if P $\neq$ NP, there is no guarantee that one-way functions exist.
- No functions have been proved to be one-way.
- Is breaking glass a one-way function?

Candidates of One-Way Functions

- Modular exponentiation $f(x) = g^x \bmod p$, where g is a primitive root of p.
  - **Discrete logarithm** is hard.[a]
- The RSA[b] function $f(x) = x^e \bmod pq$ for an odd e relatively prime to $\phi(pq)$.
  - Breaking the RSA function is hard.

---

[a] Conjectured to be 2^{n^ϵ} for some $\epsilon > 0$ in both the worst-case sense and average sense. Doable in time $n^{O(\log n)}$ for finite fields of small characteristic (Barbulescu, et al., 2013). It is in NP in some sense (Grollmann & Selman, 1988).

[b] Rivest, Shamir, & Adleman (1978).

Candidates of One-Way Functions (concluded)

- Modular squaring $f(x) = x^2 \bmod pq$.
  - Determining if a number with a Jacobi symbol 1 is a quadratic residue is hard—the **quadratic residuacity assumption** (**QRA**).[a]
  - Breaking it is as hard as factorization when $p \equiv q \equiv 3 \bmod 4$.[b]

---

[a] Due to Gauss.

[b] Rabin (1979).

The Secret-Key Agreement Problem

- Exchanging messages securely using a private-key cryptosystem requires Alice and Bob have the *same* key.[a]
  - An example is the r in the one-time pad.[b]
- How can they agree on the same secret key when the channel is insecure?
- This is called the **secret-key agreement problem**.
- It was solved by Diffie and Hellman (1976) using one-way functions.

---

[a] See p. 642.
[b] See p. 641.

The Diffie-Hellman Secret-Key Agreement Protocol

1: Alice and Bob agree on a large prime p and a primitive root g of p; $\{p$ and g are public.$\}$

2: Alice chooses a large number a at random;

3: Alice computes $\alpha = g^a \bmod p$;

4: Bob chooses a large number b at random;

5: Bob computes $\beta = g^b \bmod p$;

6: Alice sends α to Bob, and Bob sends β to Alice;

7: Alice computes her key $\beta^a \bmod p$;

8: Bob computes his key $\alpha^b \bmod p$;

Analysis

- The keys computed by Alice and Bob are identical as

$$\beta^a = g^{ba} = g^{ab} = \alpha^b \bmod p.$$

- To compute the common key from p, g, α, β is known as the **Diffie-Hellman problem**.
- It is conjectured to be hard.[a]
- If discrete logarithm is easy, then one can solve the Diffie-Hellman problem.
  - Because a and b can then be obtained by Eve.
- But the other direction is still open.

[a] This is the **computational Diffie-Hellman assumption** (CDH).

The RSA Function

- Let p, q be two distinct primes.
- The RSA function is $x^e \bmod pq$ for an odd e relatively prime to $\phi(pq)$.
  - By Lemma 59 (p. 484),

$$\phi(pq) = pq \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right) = pq - p - q + 1. \quad (15)$$

- As $\gcd(e, \phi(pq)) = 1$, there is a d such that

$$ed \equiv 1 \bmod \phi(pq),$$

which can be found by the Euclidean algorithm.[a]

[a]One can think of d as e^{-1}.

A Public-Key Cryptosystem Based on RSA

- Bob generates p and q.
- Bob publishes pq and the encryption key e, a number relatively prime to $\phi(pq)$.
  - The encryption function is
    $$y = x^e \bmod pq.$$
  - Bob calculates $\phi(pq)$ by Eq. (15) (p. 655).
  - Bob then calculates d such that $ed = 1 + k\phi(pq)$ for some $k \in \mathbb{Z}$.

A Public-Key Cryptosystem Based on RSA (continued)

- The decryption function is

$$y^d \bmod pq.$$

- It works because

$$y^d = x^{ed} = x^{1+k\phi(pq)} = x \bmod pq$$

by the Fermat-Euler theorem when $\gcd(x, pq) = 1$ (p. 489).

A Public-Key Cryptosystem Based on RSA (continued)

- What if x is not relatively prime to pq?[a]
- As $\phi(pq) = (p-1)(q-1)$,

$$ed = 1 + k(p-1)(q-1).$$

- Say $x \equiv 0 \bmod p$.
- Then

$$y^d \equiv x^{ed} \equiv 0 \equiv x \bmod p.$$

[a]Of course, one would be unlucky here.

A Public-Key Cryptosystem Based on RSA (continued)

- On the other hand, either $x \not\equiv 0 \bmod q$ or $x \equiv 0 \bmod q$.
- If $x \not\equiv 0 \bmod q$, then

$$\begin{aligned} y^d &\equiv x^{ed} \equiv x^{ed-1} x \equiv x^{k(p-1)(q-1)} x \equiv \left(x^{q-1}\right)^{k(p-1)} x \\ &\equiv x \bmod q. \end{aligned}$$

  by Fermat's "little" theorem (p. 487).

- If $x \equiv 0 \bmod q$, then

$$y^d \equiv x^{ed} \equiv 0 \equiv x \bmod q.$$

A Public-Key Cryptosystem Based on RSA (concluded)

- By the Chinese remainder theorem (p. 486),

$$y^d \equiv x^{ed} \equiv 0 \equiv x \bmod pq,$$

even when x is not relatively prime to p.

- When x is not relatively prime to q, the same conclusion holds.

The "Security" of the RSA Function

- Factoring pq or calculating d from (e, pq) seems hard.
- Breaking the last bit of RSA is as hard as breaking the RSA.[a]
- Recommended RSA key sizes:[b]
  - 1024 bits up to 2010.
  - 2048 bits up to 2030.
  - 3072 bits up to 2031 and beyond.

---

[a] Alexi, Chor, Goldreich, & Schnorr (1988).

[b] RSA (2003). RSA was acquired by EMC in 2006 for 2.1 billion US dollars.

The "Security" of the RSA Function (continued)

- Recall that problem A is "harder than" problem B if solving A results in solving B.
  - Factorization is "harder than" breaking the RSA.
  - It is not hard to show that calculating Euler's phi function[a] is "harder than" breaking the RSA.
  - Factorization is "harder than" calculating Euler's phi function (see Lemma 59 on p. 484).
  - So factorization is harder than calculating Euler's phi function, which is harder than breaking the RSA.

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[a] When the input is not factorized!

The "Security" of the RSA Function (concluded)

- Factorization cannot be NP-hard unless NP = coNP.[a]
- So breaking the RSA is unlikely to imply P = NP.
- But numbers can be factorized efficiently by quantum computers.[b]
- RSA was alleged to have received 10 million US dollars from the government to promote unsecure p and q.[c]

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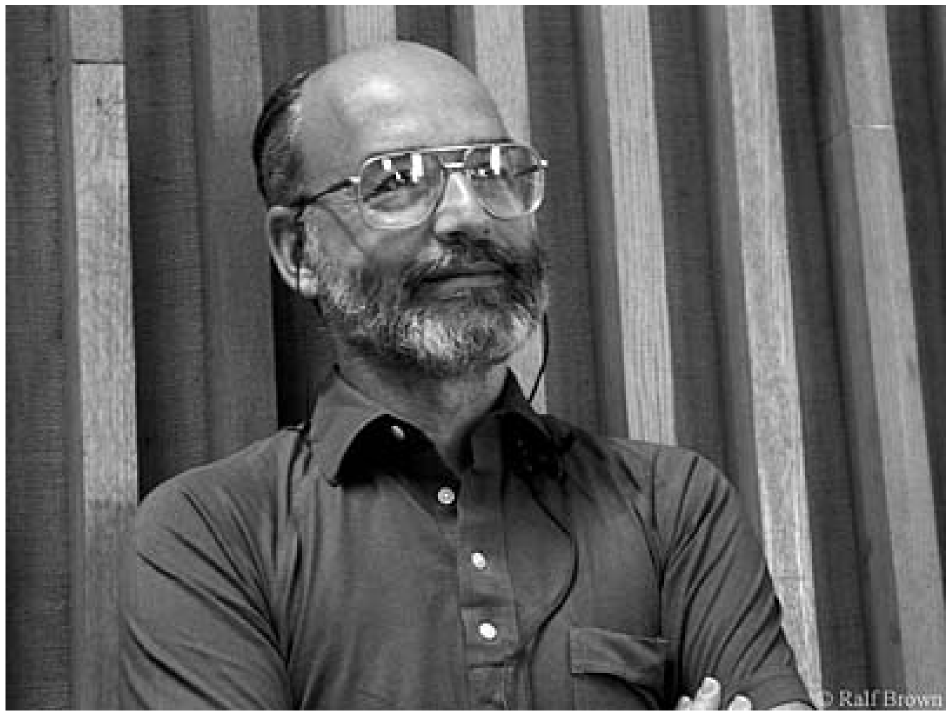
[a] Brassard (1979).
[b] Shor (1994).
[c] Menn (2013).

Adi Shamir, Ron Rivest, and Leonard Adleman

Ron Rivest[a] (1947–)

[a]Turing Award (2002).

Adi Shamir[a] (1952–)



[a]Turing Award (2002).

A Parallel History

- Diffie and Hellman's solution to the secret-key agreement problem led to public-key cryptography.
- In 1973, the RSA public-key cryptosystem was invented in Britain before the Diffie-Hellman secret-key agreement scheme.[a]

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[a]Ellis, Cocks, and Williamson of the Communications Electronics Security Group of the British Government Communications Head Quarters (GCHQ).

Is a forged signature the same sort of thing
as a genuine signature,
or is it a different sort of thing?
— Gilbert Ryle (1900–1976),
The Concept of Mind (1949)

"Katherine, I gave him the code.
He verified the code."
"But did you verify him?"
— *The Numbers Station* (2013)

Digital Signatures[a]

- Alice wants to send Bob a *signed* document x.
- The signature must unmistakably identifies the sender.
- Both Alice and Bob have public and private keys

$$e_{\text{Alice}}, e_{\text{Bob}}, d_{\text{Alice}}, d_{\text{Bob}}.$$

- Every cryptosystem guarantees $D(d, E(e, x)) = x$.
- Assume the cryptosystem also satisfies the commutative property

$$E(e, D(d, x)) = D(d, E(e, x)). \tag{16}$$

  – E.g., the RSA system satisfies it as $(x^d)^e = (x^e)^d$.

[a] Diffie & Hellman (1976).

Digital Signatures Based on Public-Key Systems

- Alice signs x as
$$(x, D(d_{\text{Alice}}, x)).$$

- Bob receives (x, y) and verifies the signature by checking
$$E(e_{\text{Alice}}, y) = E(e_{\text{Alice}}, D(d_{\text{Alice}}, x)) = x$$
based on Eq. (16).

- The claim of authenticity is founded on the difficulty of inverting E_{Alice} without knowing the key d_{Alice}.

Blind Signatures[a]

- There are applications where the document author (Alice) and the signer (Bob) are *different* parties.
- Sender privacy: We do not want Bob to see the document.
  - Anonymous electronic voting systems, digital cash schemes, anonymous payments, etc.
- Idea: The document is **blinded** by Alice before it is signed by Bob.
- The resulting blind signature can be publicly verified against the original, unblinded document x as before.

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[a]Chaum (1983).

Blind Signatures Based on RSA

Blinding by Alice:

1: Pick $r \in Z_n^*$ randomly;

2: Send

$$x' = x r^e \bmod n$$

to Bob; $\{x$ is blinded by $r^e.\}$

- Note that $r \to r^e \bmod n$ is a one-to-one correspondence.
- Hence $r^e \bmod n$ is a random number, too.
- As a result, x' is random and leaks no information, even if x has any structure.

Blind Signatures Based on RSA (continued)

Signing by Bob with his private decryption key d:

1: Send the blinded signature

$$s' = (x')^d \bmod n$$

to Alice;

Blind Signatures Based on RSA (continued)

The RSA signature of Alice:

1: Alice obtains the signature $s = s' r^{-1} \bmod n$;

- This works because

  $$s \equiv s' r^{-1} \equiv (x')^d r^{-1} \equiv (x r^e)^d r^{-1} \equiv x^d r^{ed-1} \equiv x^d \bmod n$$

  by the properties of the RSA function.

- Note that only Alice knows r.

Blind Signatures Based on RSA (concluded)

- Anyone can verify the document was signed by Bob by checking with Bob's encryption key e the following:

$$s^e \equiv x \bmod n.$$

- But Bob does not know s is related to x' (thus Alice).