Boolean Functions

- An n-ary boolean function is a function

$$f : \{\, \texttt{true}, \texttt{false} \,\}^n \to \{\, \texttt{true}, \texttt{false} \,\}.$$

- It can be represented by a truth table.
- There are 2^{2^n} such boolean functions.
  - We can assign `true` or `false` to f for each of the 2^n truth assignments.

Boolean Functions (continued)

Assignment	Truth value
1	`true` or `false`
2	`true` or `false`
$\vdots$	$\vdots$
2^n	`true` or `false`

- A boolean expression expresses a boolean function.
  - Think of its truth values under all possible truth assignments.

Boolean Functions (continued)

- A boolean function expresses a boolean expression.
  - $\bigvee_{T \,\models\, \phi,\ \text{literal } y_i \text{ is true in "row" } T} (y_1 \wedge \cdots \wedge y_n)$.[a]
    - The implicant $y_1 \wedge \cdots \wedge y_n$ is called the **minterm** over $\{ x_1, \ldots, x_n \}$ for T.
  - The size[b] is $\leq n2^n \leq 2^{2n}$.
  - This DNF is optimal for the parity function, for example.[c]

---

[a] Similar to **programmable logic array**. This is called the **table lookup representation** (Beigel, 1993).

[b] We count only the literals here.

[c] Du & Ko (2000).

Boolean Functions (continued)

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

The corresponding boolean expression:

$$(\neg x_1 \land \neg x_2) \lor (\neg x_1 \land x_2) \lor (x_1 \land x_2).$$

Boolean Functions (concluded)

Corollary 15 *Every n-ary boolean function can be expressed by a boolean expression of size $O(n2^n)$.*

- In general, the exponential length in n cannot be avoided (p. 209).
- The size of the truth table is also $O(n2^n)$.[a]

[a] There are 2^n n-bit strings.

Boolean Circuits

- A **boolean circuit** is a graph C whose nodes are the **gates**.
- There are no cycles in C.
- All nodes have indegree (number of incoming edges) equal to 0, 1, or 2.
- Each gate has a **sort** from

$$\{\texttt{true}, \texttt{false}, \vee, \wedge, \neg, x_1, x_2, \ldots\}.$$

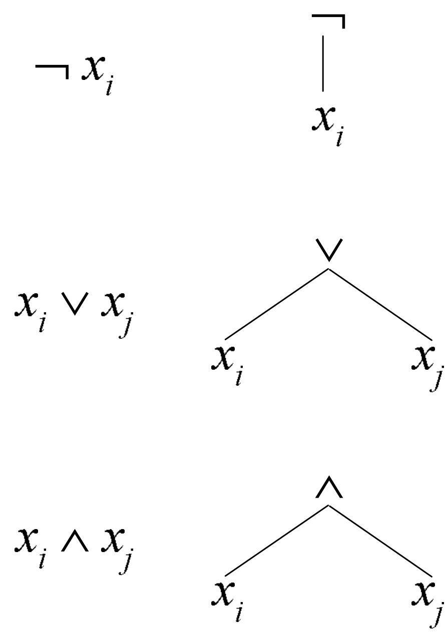
  - There are $n + 5$ sorts.

Boolean Circuits (concluded)

- Gates with a sort from $\{\texttt{true}, \texttt{false}, x_1, x_2, \ldots\}$ are the **inputs** of C and have an indegree of zero.
- The **output gate**(s) has no outgoing edges.
- A boolean circuit computes a boolean function.
- A boolean function can be realized by *infinitely many* equivalent boolean circuits.

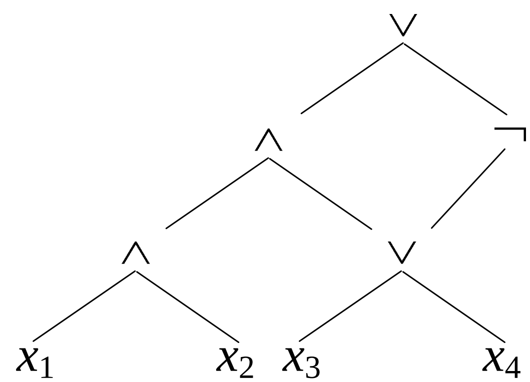
Boolean Circuits and Expressions

- They are equivalent representations.
- One can construct one from the other:



An Example

$$((x_1 \wedge x_2) \wedge (x_3 \vee x_4)) \vee (\neg(x_3 \vee x_4))$$



- Circuits are more economical because of the possibility of "sharing."

CIRCUIT SAT and CIRCUIT VALUE

CIRCUIT SAT: Given a circuit, is there a truth assignment such that the circuit outputs true?

- CIRCUIT SAT $\in$ NP: Guess a truth assignment and then evaluate the circuit.[a]

CIRCUIT VALUE: The same as CIRCUIT SAT except that the circuit has no variable gates.

- CIRCUIT VALUE $\in$ P: Evaluate the circuit from the input gates gradually towards the output gate.

---

[a]Essentially the same algorithm as the one on p. 119.

Some[a] Boolean Functions Need Exponential Circuits[b]

Theorem 16 *For any $n \geq 2$, there is an n-ary boolean function f such that no boolean circuits with $2^n/(2n)$ or fewer gates can compute it.*

- There are 2^{2^n} different n-ary boolean functions (p. 199).
- So it suffices to prove that the number of boolean circuits with $2^n/(2n)$ or fewer gates is less than 2^{2^n}.

---

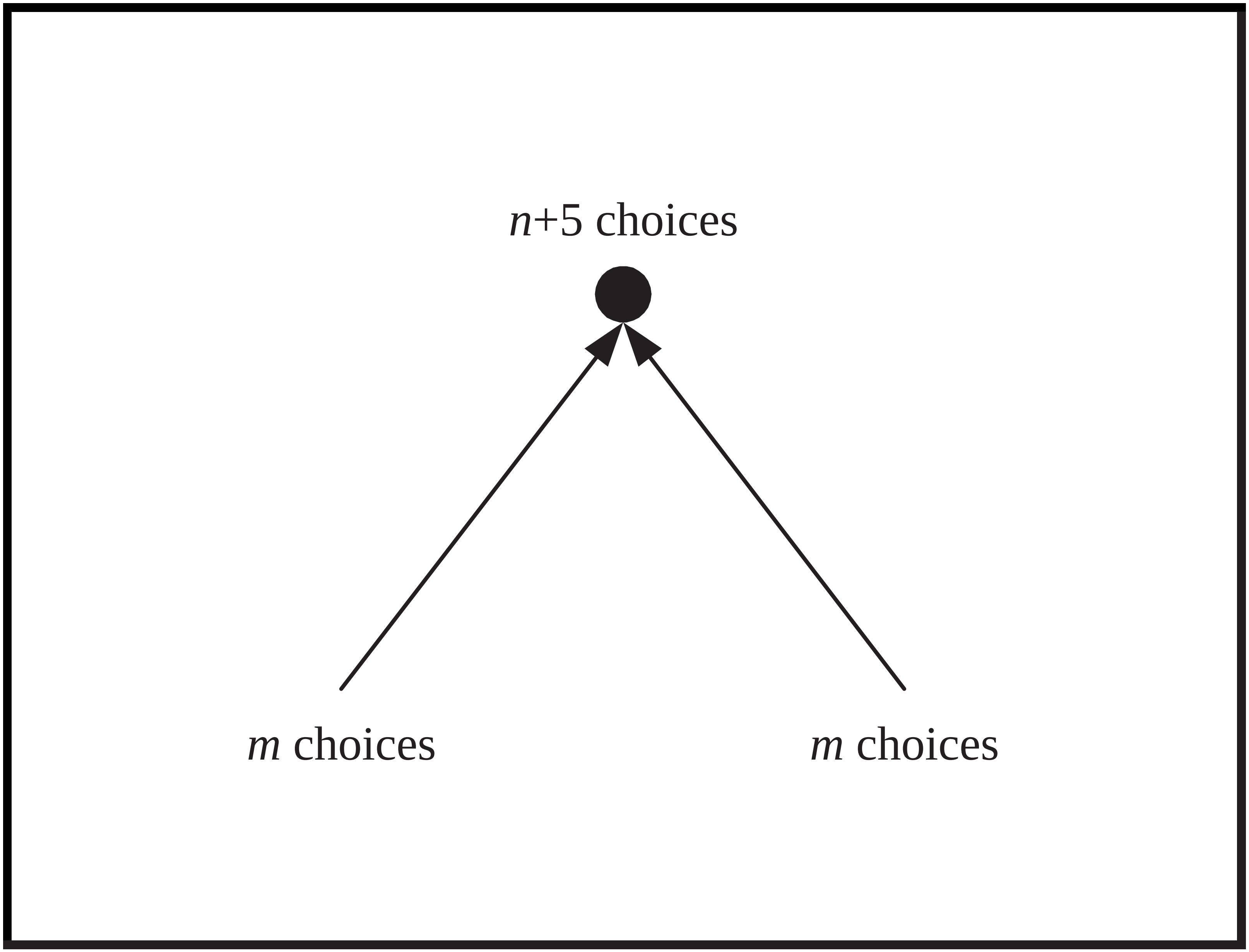
[a]Can be strengthened to "Almost all."
[b]Riordan & Shannon (1942); Shannon (1949).

The Proof (concluded)

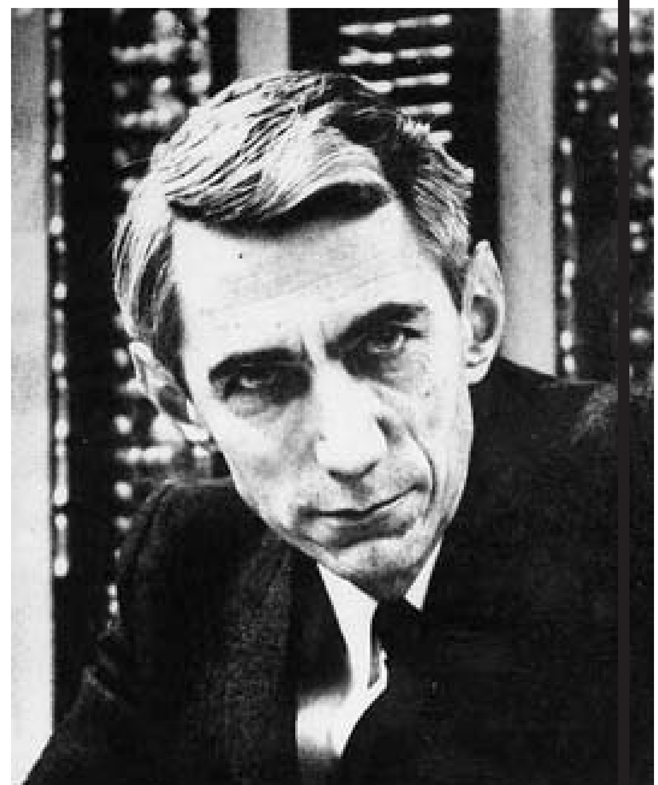
- There are at most $((n+5) \times m^2)^m$ boolean circuits with m or fewer gates (see next page).
- But $((n+5) \times m^2)^m < 2^{2^n}$ when $m = 2^n/(2n)$:

$$
\begin{aligned}
& m \log_2((n+5) \times m^2) \\
= \;& 2^n \left(1 - \frac{\log_2 \frac{4n^2}{n+5}}{2n} \right) \\
< \;& 2^n
\end{aligned}
$$

for $n \geq 2$.


n+5 choices
m choices
m choices

Claude Elwood Shannon (1916–2001)



Howard Gardner (1987), "[Shannon's master's thesis is] possibly the most important, and also the most famous, master's thesis of the century."

Comments

- The lower bound $2^n/(2n)$ is rather tight because an upper bound is $n2^n$ (p. 201).
- The proof counted the number of circuits.
  - Some circuits may not be valid at all.
  - Different circuits may also compute the same function.
- Both are fine because we only need an upper bound on the number of circuits.
- We do not need to consider the *outgoing* edges because they have been counted as incoming edges.[a]

[a]If you prove it by considering outgoing edges, the bound will not be good. (Try it!)

Relations between Complexity Classes

It is, I own, not uncommon to be
wrong in theory
and right in practice.
— Edmund Burke (1729–1797),
A Philosophical Enquiry into the Origin of Our Ideas of the Sublime and Beautiful (1757)

The problem with QE is
it works in practice,
but it doesn't work in theory.
— Ben Bernanke (2014)

Proper (Complexity) Functions

- We say that $f : \mathbb{N} \to \mathbb{N}$ is a **proper (complexity) function** if the following hold:
  - f is nondecreasing.
  - There is a k-string TM M_f such that $M_f(x) = \sqcap^{f(|x|)}$ for any x.[a]
  - M_f halts after $O(|x| + f(|x|))$ steps.
  - M_f uses $O(f(|x|))$ space besides its input x.
- M_f's behavior depends only on $|x|$ not x's contents.
- M_f's running time is bounded by $f(n)$.

[a]The textbook calls "$\sqcap$" the quasi-blank symbol. The use of $M_f(x)$ will become clear in Proposition 17 (p. 219).

Examples of Proper Functions

- Most "reasonable" functions are proper: c, $\lceil \log n \rceil$, polynomials of n, 2^n, $\sqrt{n}$, $n!$, etc.
- If f and g are proper, then so are $f + g$, fg, and 2^g.[a]
- Nonproper functions when serving as the time bounds for complexity classes spoil "theory building."
  - For example, $\text{TIME}(f(n)) = \text{TIME}(2^{f(n)})$ for some recursive function f (the **gap theorem**).[b]
- Only proper functions f will be used in $\text{TIME}(f(n))$, $\text{SPACE}(f(n))$, $\text{NTIME}(f(n))$, and $\text{NSPACE}(f(n))$.

[a] For $f(g(n))$, we need to add $f(n) \geq n$.

[b] Trakhtenbrot (1964); Borodin (1972). Theorem 7.3 on p. 145 of the textbook proves it.

Precise Turing Machines

- A TM M is **precise** if there are functions f and g such that for every $n \in \mathbb{N}$, for every x of length n, and for every computation path of M,
  - M halts after precisely $f(n)$ steps,[a] and
  - All of its strings are of length precisely $g(n)$ at halting.[b]
    - Recall that if M is a TM with input and output, we exclude the first and last strings.
- M can be deterministic or nondeterministic.

---

[a] Fully time constructible (Hopcroft & Ullman, 1979).
[b] Fully space constructible (Hopcroft & Ullman, 1979).

Precise TMs Are General

Proposition 17 *Suppose a TM*[a] *M decides L within time (space) $f(n)$, where f is proper. Then there is a precise TM M' which decides L in time $O(n + f(n))$ (space $O(f(n))$, respectively).*

- M' on input x first simulates the TM M_f associated with the proper function f on x.
- M_f's output, of length $f(|x|)$, will serve as a "yardstick" or an "alarm clock."

---

[a] It can be deterministic or nondeterministic.

The Proof (continued)

- Then M' simulates $M(x)$.
- $M'(x)$ halts when and only when the alarm clock runs out—even if M halts earlier.
- If f is a time bound:
  - The simulation of each step of M on x is matched by advancing the cursor on the "clock" string.
  - Because M' stops at the moment the "clock" string is exhausted—even if $M(x)$ stops earlier, it is precise.
  - The time bound is therefore $O(|\,x\,| + f(|\,x\,|))$.

The Proof (concluded)

- If f is a space bound (sketch):
  - M' simulates M *on* the quasi-blanks of M_f's output string.[a]
  - The total space, not counting the input string, is $O(f(n))$.
  - But we still need a way to make sure there is no infinite loop.[b]

---

[a] This is to make sure the space bound is precise.

[b] See the proof of Theorem 24 (p. 237).

Important Complexity Classes

- We write expressions like n^k to denote the union of all complexity classes, one for each value of k.
- For example,

$$\mathrm{NTIME}(n^k) \stackrel{\Delta}{=} \bigcup_{j>0} \mathrm{NTIME}(n^j).$$

Important Complexity Classes (concluded)

$$
\begin{aligned}
\text{P} &\triangleq \text{TIME}(n^k), \\
\text{NP} &\triangleq \text{NTIME}(n^k), \\
\text{PSPACE} &\triangleq \text{SPACE}(n^k), \\
\text{NPSPACE} &\triangleq \text{NSPACE}(n^k), \\
\text{E} &\triangleq \text{TIME}(2^{kn}), \\
\text{EXP} &\triangleq \text{TIME}(2^{n^k}), \\
\text{NEXP} &\triangleq \text{NTIME}(2^{n^k}), \\
\text{L} &\triangleq \text{SPACE}(\log n), \\
\text{NL} &\triangleq \text{NSPACE}(\log n).
\end{aligned}
$$

Complements of Nondeterministic Classes

- Recall that the complement of L, or $\bar{L}$, is the language $\Sigma^* - L$.
  - SAT COMPLEMENT is the set of unsatisfiable boolean expressions.
- R, RE, and coRE are distinct (p. 158).
  - Again, coRE contains the complements of *languages* in RE, *not* languages that are not in RE.
- How about co$\mathcal{C}$ when $\mathcal{C}$ is a complexity class?

The Co-Classes

- For any complexity class $\mathcal{C}$, co$\mathcal{C}$ denotes the class

$$\{ L : \bar{L} \in \mathcal{C} \}.$$

- Clearly, if $\mathcal{C}$ is a *deterministic* time or space *complexity class*, then $\mathcal{C} = \text{co}\mathcal{C}$.
  - They are said to be **closed under complement**.
  - A deterministic TM deciding L can be converted to one that decides $\bar{L}$ within the same time or space bound by reversing the "yes" and "no" states.[a]
- Whether *non*deterministic classes for time are closed under complement is not known (see p. 111).

---

[a]See p. 155.

Comments

- As
$$\text{co}\mathcal{C} = \{\, L : \bar{L} \in \mathcal{C} \,\},$$
$L \in \mathcal{C}$ if and only if $\bar{L} \in \text{co}\mathcal{C}$.
- But it is *not* true that $L \in \mathcal{C}$ if and only if $L \not\in \text{co}\mathcal{C}$.
  - co$\mathcal{C}$ is not defined as $\bar{\mathcal{C}}$.
- For example, suppose $\mathcal{C} = \{\{\, 2, 4, 6, 8, 10, \ldots \,\}, \ldots \,\}$.
- Then co$\mathcal{C} = \{\{\, 1, 3, 5, 7, 9, \ldots \,\}, \ldots \,\}$.
- But $\bar{\mathcal{C}} = 2^{\{\, 1,2,3,\ldots \,\}} - \{\{\, 2, 4, 6, 8, 10, \ldots \,\}, \ldots \,\}$.

The Quantified Halting Problem

- Let $f(n) \geq n$ be proper.
- Define

$$H_f \stackrel{\Delta}{=} \{\, M; x : M \text{ accepts input } x \text{ after at most } f(\,|\,x\,|\,) \text{ steps} \,\},$$

where M is deterministic.

- Assume the input is binary as usual.

$H_f \in \mathsf{TIME}(f(n)^3)$

- For each input $M; x$, we simulate M on x with an alarm clock of length $f(|\,x\,|)$.
  - Use the single-string simulator (p. 83), the universal TM (p. 135), and the linear speedup theorem (p. 93).
  - Our simulator accepts $M; x$ if and only if M accepts x before the alarm clock runs out.
- From p. 90, the total running time is $O(\ell_M k_M^2 f(n)^2)$, where ℓ_M is the length to encode each symbol or state of M and k_M is M's number of strings.
- As $\ell_M k_M^2 = O(n)$, the running time is $O(f(n)^3)$, where the constant is independent of M.

$H_f \notin \mathsf{TIME}(f(\lfloor n/2 \rfloor))$

- Suppose TM M_{H_f} decides H_f in time $f(\lfloor n/2 \rfloor)$.
- Consider machine:

$$
\begin{aligned}
D_f(M) \quad \{ & \\
& \textbf{if } M_{H_f}(M;M) = \text{“yes”} \\
& \textbf{then } \text{“no”}; \\
& \textbf{else } \text{“yes”}; \\
\} &
\end{aligned}
$$

- D_f on input M runs in the same time as M_{H_f} on input $M;M$, i.e., in time $f(\lfloor \frac{2n+1}{2} \rfloor) = f(n)$, where $n = |\, M \,|$.[a]

[a]A student pointed out on October 6, 2004, that this estimation forgets to include the time to write down $M;M$.

The Proof (concluded)

- First,

$$\begin{aligned} & D_f(D_f) = \text{“yes”} \\ \Rightarrow \quad & D_f; D_f \notin H_f \\ \Rightarrow \quad & D_f \text{ does not accept } D_f \text{ within time } f(|\, D_f \,|) \\ \Rightarrow \quad & D_f(D_f) = \text{“no”} \quad \text{as } D_f(D_f) \text{ runs in time } f(|\, D_f \,|), \end{aligned}$$

  a contradiction

- Similarly, $D_f(D_f) =$ “no” $\Rightarrow D_f(D_f) =$ “yes.”

The Time Hierarchy Theorem

Theorem 18 *If $f(n) \geq n$ is proper, then*

$$\text{TIME}(f(n)) \subsetneq \text{TIME}(f(2n+1)^3).$$

- The quantified halting problem makes it so.

Corollary 19 $\text{P} \subsetneq \text{E}$.

- $\text{P} \subseteq \text{TIME}(2^n)$ because $\text{poly}(n) \leq 2^n$ for n large enough.
- But by Theorem 18,

$$\text{TIME}\,(2^n) \subsetneq \text{TIME}\left((2^{2n+1})^3\right) \subseteq \text{E}.$$

- So $\text{P} \subsetneq \text{E}$.

The Space Hierarchy Theorem

Theorem 20 (Hennie & Stearns, 1966) *If $f(n)$ is proper, then*

$$\mathrm{SPACE}(f(n)) \subsetneq \mathrm{SPACE}(f(n) \log f(n)).$$

Corollary 21 $\mathrm{L} \subsetneq \mathrm{PSPACE}$.

Nondeterministic Time Hierarchy Theorems

Theorem 22 (Cook, 1973) $\mathrm{NTIME}(n^r) \subsetneq \mathrm{NTIME}(n^s)$ *whenever* $1 \leq r < s$.

Theorem 23 (Seiferas, Fischer, & Meyer, 1978) *If* $T_1(n)$ *and* $T_2(n)$ *are proper, then*

$$\mathrm{NTIME}(T_1(n)) \subsetneq \mathrm{NTIME}(T_2(n))$$

whenever $T_1(n+1) = o(T_2(n))$.

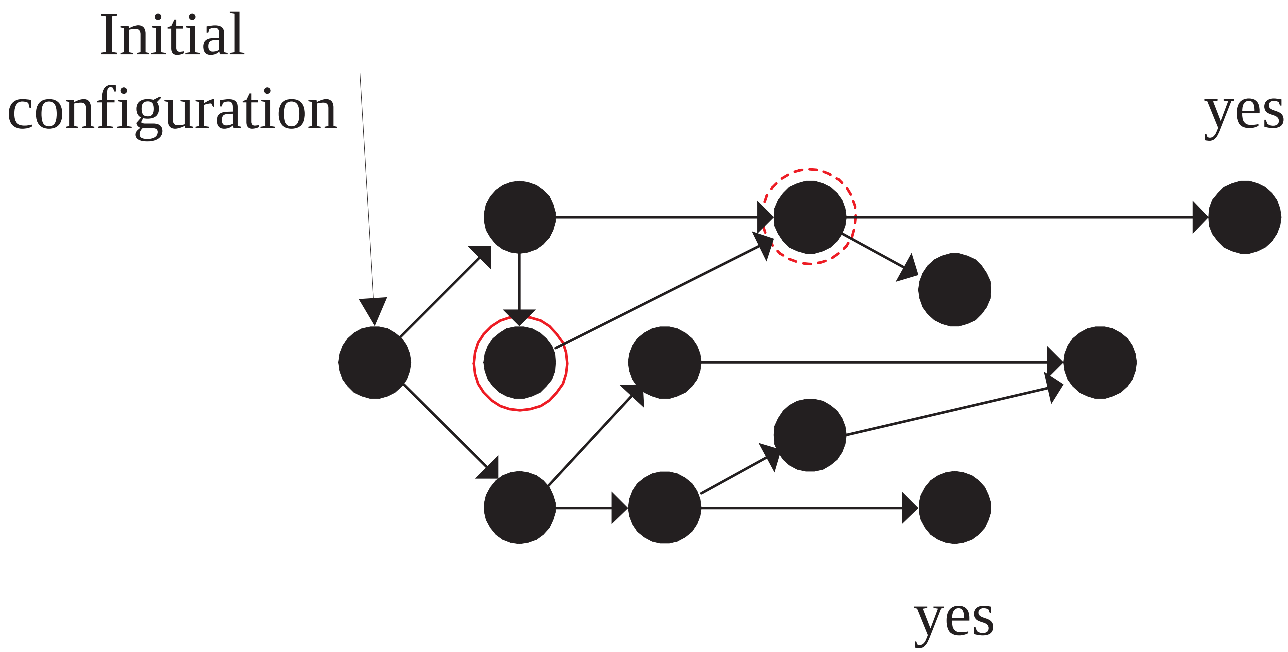
The Reachability Method

- The computation of a time-bounded TM can be represented by a directed graph.
- The TM's configurations constitute the nodes.
- There is a directed edge from node x to node y if x yields y in one step.
- The start node representing the initial configuration has zero in-degree.

The Reachability Method (concluded)

- When the TM is nondeterministic, a node may have an out-degree greater than one.
  - The graph is the same as the computation tree earlier.
  - But identical configurations are merged into one node.
- So M accepts the input if and only if there is a path from the start node to a node with a "yes" state.
- It is the reachability problem.

Illustration of the Reachability Method



Relations between Complexity Classes

Theorem 24 *Suppose $f(n)$ is proper. Then*

1. $\text{SPACE}(f(n)) \subseteq \text{NSPACE}(f(n))$,
   $\text{TIME}(f(n)) \subseteq \text{NTIME}(f(n))$.
2. $\text{NTIME}(f(n)) \subseteq \text{SPACE}(f(n))$.
3. $\text{NSPACE}(f(n)) \subseteq \text{TIME}(k^{\log n + f(n)})$.

- Proof of 2:
  - Explore the computation *tree* of the NTM for "yes."
  - Specifically, generate an $f(n)$-bit sequence denoting the nondeterministic choices over $f(n)$ steps.

Proof of Theorem 24(2)

- (continued)
  - Simulate the NTM based on the choices.
  - Recycle the space and repeat the above steps.
  - Halt with "yes" when a "yes" is encountered or "no" if the tree is exhausted.
  - Each path simulation consumes at most $O(f(n))$ space because it takes $O(f(n))$ time.
  - The total space is $O(f(n))$ because space is recycled.

Proof of Theorem 24(3)

- Let k-string NTM

$$M = (K, \Sigma, \Delta, s)$$

with input and output decide $L \in \text{NSPACE}(f(n))$.

- Use the reachability method on the configuration graph of M on input x of length n.

- A configuration is a $(2k+1)$-tuple

$$(q, w_1, u_1, w_2, u_2, \ldots, w_k, u_k).$$

Proof of Theorem 24(3) (continued)

- We only care about

$$(q, i, w_2, u_2, \ldots, w_{k-1}, u_{k-1}),$$

  where i is an integer between 0 and n for the position of the *first* cursor.

- The number of configurations is therefore at most

$$|\,K\,| \times (n+1) \times |\,\Sigma\,|^{2(k-2)f(n)} = O(c_1^{\log n + f(n)}) \tag{2}$$

  for some $c_1 > 1$, which depends on M.

- Add edges to the configuration graph based on M's transition function.

Proof of Theorem 24(3) (concluded)

- $x \in L \Leftrightarrow$ there is a path in the configuration graph from the initial configuration to a configuration of the form ("yes", $i, \ldots$).[a]
- This is REACHABILITY on a graph with $O(c_1^{\log n + f(n)})$ nodes.
- It is in TIME$(c^{\log n + f(n)})$ for some $c > 1$ because REACHABILITY $\in$ TIME(n^j) for some j and

$$\left[c_1^{\log n + f(n)} \right]^j = (c_1^j)^{\log n + f(n)}.$$

[a]There may be many of them.

Space-Bounded Computation and Proper Functions

- In the definition of *space-bounded* computations earlier (p. 109), the TMs are not required to halt at all.
- When the space is bounded by a proper function f, computations can be assumed to halt:
  - Run the TM associated with f to produce a quasi-blank output of length $f(n)$ first.
  - The space-bounded computation must repeat a configuration if it runs for more than $c^{\log n+f(n)}$ steps for some $c > 1$.[a]

[a] See Eq. (2) on p. 240.

Space-Bounded Computation and Proper Functions (concluded)

- (continued)
  - So an infinite loop occurs during simulation for a computation path longer than $c^{\log n + f(n)}$ steps.
  - Hence we only simulate up to $c^{\log n + f(n)}$ time steps per computation path.

A Grand Chain of Inclusions[a]

- It is an easy application of Theorem 24 (p. 237) that

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq EXP.$$

- By Corollary 21 (p. 232), we know $L \subsetneq PSPACE$.
- So the chain must break somewhere between L and EXP.
- It is suspected that all four inclusions are proper.
- But there are no proofs yet.

[a]With input from Mr. Chin-Luei Chang (B89902053, R93922004, D95922007) on October 22, 2004.

What Is Wrong with the Proof?[a]

- By Theorem 24(2) (p. 237),

$$\text{NL} \subseteq \text{TIME}\left(k^{O(\log n)}\right) \subseteq \text{TIME}\,(n^{c_1})$$

  for some $c_1 > 0$.

- By Theorem 18 (p. 231),

$$\text{TIME}\,(n^{c_1}) \subsetneq \text{TIME}\,(n^{c_2}) \subseteq \text{P}$$

  for some $c_2 > c_1$.

- So

$$\text{NL} \neq \text{P}.$$

[a]Contributed by Mr. Yuan-Fu Shao (R02922083) on November 11, 2014.

What Is Wrong with the Proof? (concluded)

- Recall from p. 222 that $\text{TIME}(k^{O(\log n)})$ is a shorthand for
$$\bigcup_{j>0} \text{TIME}\left(j^{O(\log n)}\right).$$
- So the correct proof runs more like
$$\text{NL} \subseteq \bigcup_{j>0} \text{TIME}\left(j^{O(\log n)}\right) \subseteq \bigcup_{c>0} \text{TIME}\left(n^c\right) = \text{P}.$$
- And
$$\text{NL} \neq \text{P}$$
no longer follows.

Nondeterministic and Deterministic Space

- By Theorem 6 (p. 116),

$$\text{NTIME}(f(n)) \subseteq \text{TIME}(c^{f(n)}),$$

  an exponential gap.
- There is no proof yet that the exponential gap is inherent.
- How about NSPACE vs. SPACE?
- Surprisingly, the relation is only quadratic—a polynomial—by Savitch's theorem.

Savitch's Theorem

Theorem 25 (Savitch, 1970)

$$\text{REACHABILITY} \in \text{SPACE}(\log^2 n).$$

- Let $G(V, E)$ be a graph with n nodes.
- For $i \geq 0$, let
$$\text{PATH}(x, y, i)$$
mean there is a path from node x to node y of length at most 2^i.
- There is a path from x to y if and only if
$$\text{PATH}(x, y, \lceil \log n \rceil)$$
holds.

The Proof (continued)

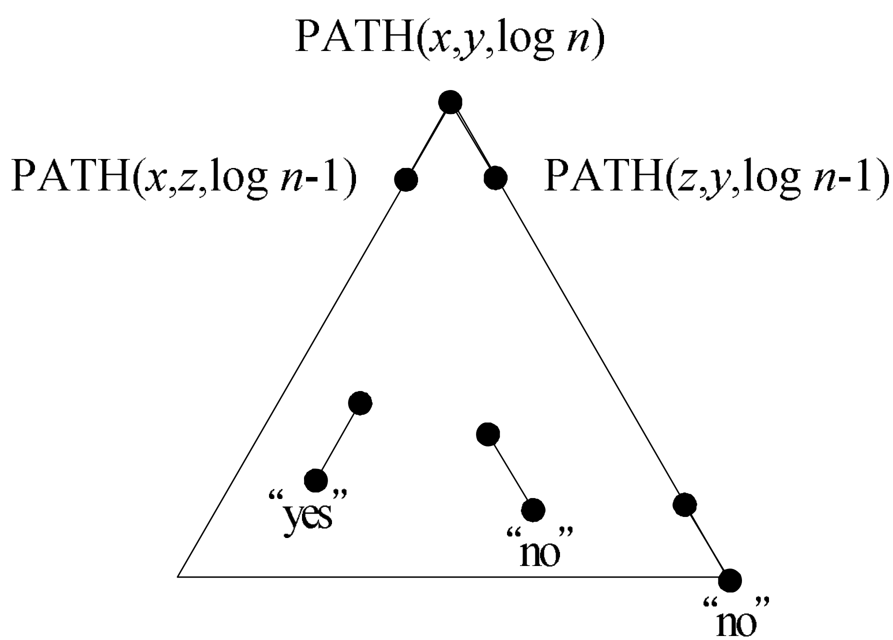
- For $i > 0$, PATH(x, y, i) if and only if there exists a z such that PATH$(x, z, i-1)$ and PATH$(z, y, i-1)$.
- For PATH$(x, y, 0)$, check the input graph or if $x = y$.
- Compute PATH$(x, y, \lceil \log n \rceil)$ with a depth-first search on a graph with nodes (x, y, i)s (see next page).[a]
- Like stacks in recursive calls, we keep only the current path's (x, y, i)s.
- The space requirement is proportional to the depth of the tree ($\lceil \log n \rceil$) times the size of the items stored at each node.

[a]Contributed by Mr. Chuan-Yao Tan on October 11, 2011.

The Proof (continued): Algorithm for PATH(x, y, i)

```
if i = 0 then
    if x = y or (x, y) ∈ E then
        return true;
    else
        return false;
    end if
else
    for z = 1, 2, ..., n do
        if PATH(x, z, i − 1) and PATH(z, y, i − 1) then
            return true;
        end if
    end for
    return false;
end if
```

The Proof (continued)



The Proof (concluded)

- Depth is $\lceil \log n \rceil$, and each node (x, y, i) needs space $O(\log n)$.
- The total space is $O(\log^2 n)$.