Basic Modular Arithmetics[a]

- Let $m, n \in \mathbb{Z}^+$.
- $m \mid n$ means m divides n; m is n's **divisor**.
- We call the numbers $0, 1, \ldots, n-1$ the **residue** modulo n.
- The **greatest common divisor** of m and n is denoted $\gcd(m, n)$.
- The r in Theorem 51 (p. 448) is a primitive root of p.

[a]Carl Friedrich Gauss.

Basic Modular Arithmetics (concluded)

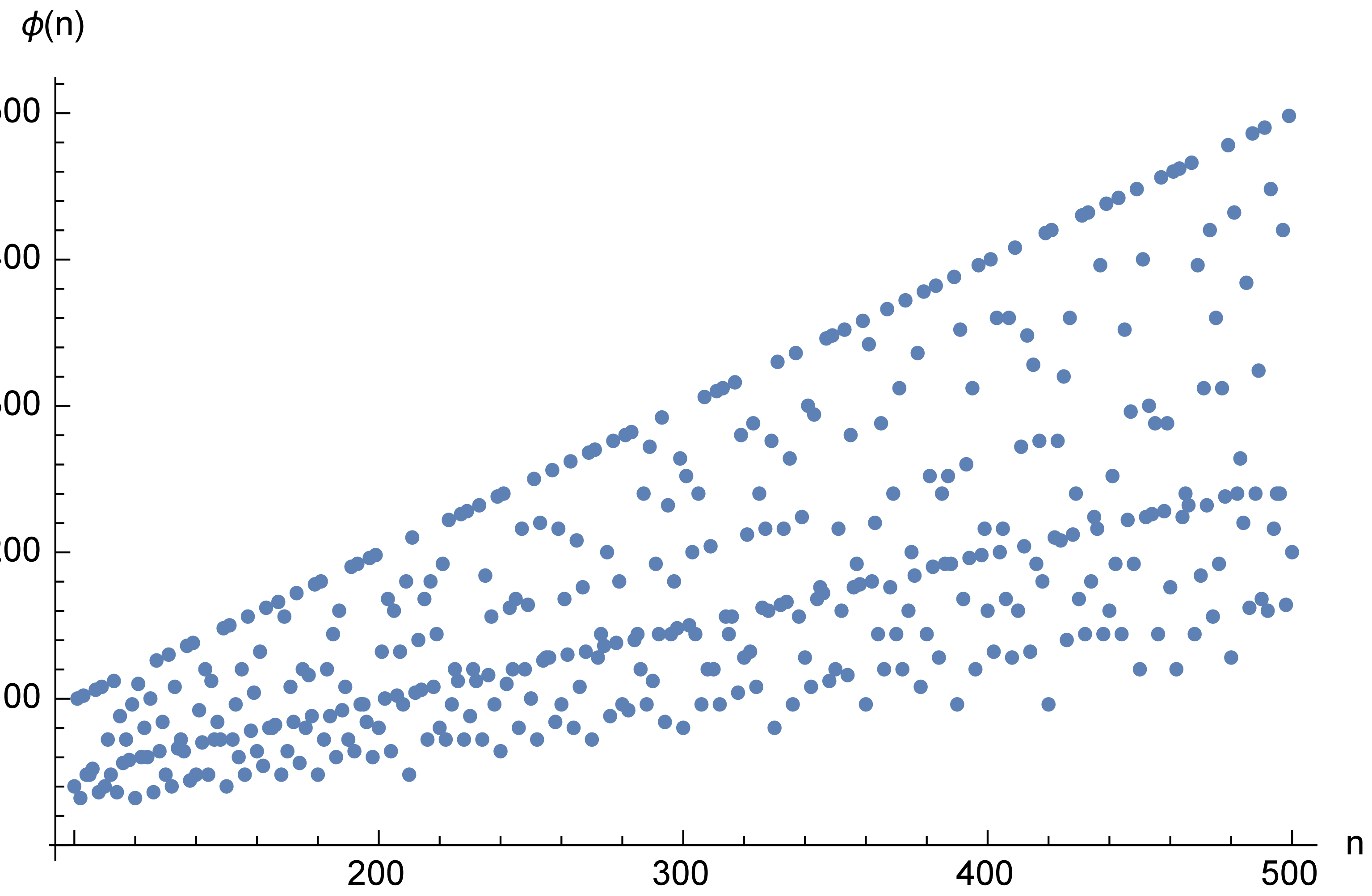
- We use
$$a \equiv b \mod n$$
if $n \mid (a - b)$.
  - So $25 \equiv 38 \bmod 13$.
- We use
$$a = b \bmod n$$
if b is the remainder of a divided by n.
  - So $25 = 12 \bmod 13$.

Euler's[a] Totient or Phi Function

- Let
  $$\Phi(n) = \{\, m : 1 \leq m < n, \gcd(m, n) = 1 \,\}$$
  be the set of all positive integers less than n that are prime to n.[b]
  - $\Phi(12) = \{\, 1, 5, 7, 11 \,\}$.
- Define **Euler's function** of n to be $\phi(n) = |\, \Phi(n) \,|$.
- $\phi(p) = p - 1$ for prime p, and $\phi(1) = 1$ by convention.
- Euler's function is not expected to be easy to compute without knowing n's factorization.

[a]Leonhard Euler (1707–1783).

[b]Z_n^* is an alternative notation.



ϕ(n)
500
400
300
200
100
200
300
400
500
n

Leonhard Euler (1707–1783)

Three Properties of Euler's Function

The inclusion-exclusion principle[a] can be used to prove the following.

Lemma 54 $\phi(n) = n \prod_{p \mid n} (1 - \frac{1}{p})$.

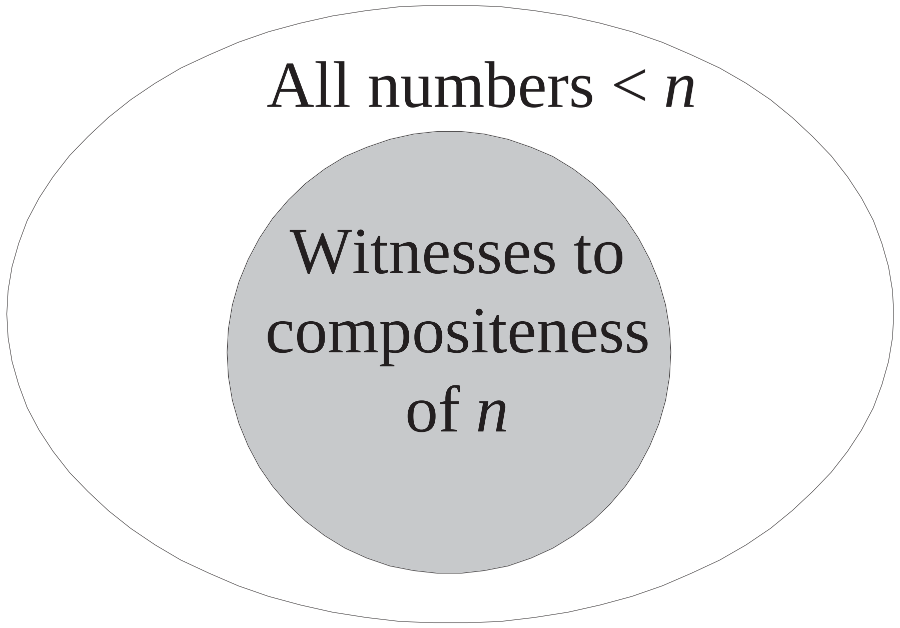
- If $n = p_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell}$ is the prime factorization of n, then
$$\phi(n) = n \prod_{i=1}^{\ell} \left(1 - \frac{1}{p_i}\right).$$

Corollary 55 $\phi(mn) = \phi(m)\,\phi(n)$ *if* $\gcd(m, n) = 1$.

Lemma 56 $\sum_{m \mid n} \phi(m) = n$.

[a]Consult any textbooks on discrete mathematics.

The Density Attack for PRIMES



The Density Attack for PRIMES

```
1: Pick $k \in \{1, \ldots, n\}$ randomly;
2: **if** $k \mid n$ and $k \neq 1$ and $k \neq n$ **then**
3:     **return** "$n$ is composite";
4: **else**
5:     **return** "$n$ is (probably) a prime";
6: **end if**
```

The Density Attack for PRIMES (continued)

- It works, but does it work well?
- The ratio of numbers $\leq n$ relatively prime to n (the white ring) is
$$\frac{\phi(n)}{n}.$$
- When $n = pq$, where p and q are distinct primes,
$$\frac{\phi(n)}{n} = \frac{pq - p - q + 1}{pq} > 1 - \frac{1}{q} - \frac{1}{p}.$$

The Density Attack for PRIMES (concluded)

- So the ratio of numbers $\leq n$ *not* relatively prime to n (the gray area) is $< (1/q) + (1/p)$.
  - The "density attack" has probability about $2/\sqrt{n}$ of factoring $n = pq$ when $p \sim q = O(\sqrt{n}\,)$.
  - The "density attack" to factor $n = pq$ hence takes $\Omega(\sqrt{n})$ steps on average when $p \sim q = O(\sqrt{n}\,)$.
  - This running time is exponential: $\Omega(2^{0.5 \log_2 n})$.

The Chinese Remainder Theorem

- Let $n = n_1 n_2 \cdots n_k$, where n_i are pairwise relatively prime.
- For any integers $a_1, a_2, \ldots, a_k$, the set of simultaneous equations

$$\begin{array}{rcl} x & = & a_1 \bmod n_1, \\ x & = & a_2 \bmod n_2, \\ & & \vdots \\ x & = & a_k \bmod n_k, \end{array}$$

has a unique solution modulo n for the unknown x.

Fermat's "Little" Theorem[a]

Lemma 57 *For all* $0 < a < p$, $a^{p-1} = 1 \bmod p$.

- Recall $\Phi(p) = \{ 1, 2, \ldots, p-1 \}$.
- Consider $a\Phi(p) = \{ am \bmod p : m \in \Phi(p) \}$.
- $a\Phi(p) = \Phi(p)$.
  - $a\Phi(p) \subseteq \Phi(p)$ as a remainder must be between 1 and $p-1$.
  - Suppose $am \equiv am' \bmod p$ for $m > m'$, where $m, m' \in \Phi(p)$.
  - That means $a(m-m') = 0 \bmod p$, and p divides a or $m - m'$, which is impossible.

[a]Pierre de Fermat (1601–1665).

The Proof (concluded)

- Multiply all the numbers in $\Phi(p)$ to yield $(p-1)!$.
- Multiply all the numbers in $a\Phi(p)$ to yield $a^{p-1}(p-1)!$.
- As $a\Phi(p) = \Phi(p)$, we have

$$a^{p-1}(p-1)! \equiv (p-1)! \bmod p.$$

- Finally, $a^{p-1} = 1 \bmod p$ because $p \not| (p-1)!$.

The Fermat-Euler Theorem[a]

Corollary 58 *For all $a \in \Phi(n)$, $a^{\phi(n)} = 1 \bmod n$.*

- The proof is similar to that of Lemma 57 (p. 470).
- Consider $a\Phi(n) = \{\, am \bmod n : m \in \Phi(n) \,\}$.
- $a\Phi(n) = \Phi(n)$.
  - $a\Phi(n) \subseteq \Phi(n)$ as a remainder must be between 0 and $n-1$ and relatively prime to n.
  - Suppose $am \equiv am' \bmod n$ for $m' < m < n$, where $m, m' \in \Phi(n)$.
  - That means $a(m - m') = 0 \bmod n$, and n divides a or $m - m'$, which is impossible.

[a] Proof by Mr. Wei-Cheng Cheng (`R93922108`, `D95922011`) on November 24, 2004.

The Proof (concluded)[a]

- Multiply all the numbers in $\Phi(n)$ to yield $\prod_{m\in\Phi(n)} m$.
- Multiply all the numbers in $a\Phi(n)$ to yield $a^{\phi(n)} \prod_{m\in\Phi(n)} m$.
- As $a\Phi(n) = \Phi(n)$,

$$\prod_{m\in\Phi(n)} m \equiv a^{\phi(n)} \left(\prod_{m\in\Phi(n)} m \right) \bmod n.$$

- Finally, $a^{\phi(n)} = 1 \bmod n$ because $n \not| \prod_{m\in\Phi(n)} m$.

[a]Some typographical errors corrected by Mr. Jung-Ying Chen (`D95723006`) on November 18, 2008.

An Example

- As $12 = 2^2 \times 3$,

$$\phi(12) = 12 \times \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 4.$$

- In fact, $\Phi(12) = \{ 1, 5, 7, 11 \}$.
- For example,

$$5^4 = 625 = 1 \bmod 12.$$

Exponents

- The **exponent** of $m \in \Phi(p)$ is the least $k \in \mathbb{Z}^+$ such that
$$m^k = 1 \bmod p.$$
- Every residue $s \in \Phi(p)$ has an exponent.
  - $1, s, s^2, s^3, \ldots$ eventually repeats itself modulo p, say $s^i \equiv s^j \bmod p$, which means $s^{j-i} = 1 \bmod p$.
- If the exponent of m is k and $m^\ell = 1 \bmod p$, then $k \mid \ell$.
  - Otherwise, $\ell = qk + a$ for $0 < a < k$, and $m^\ell = m^{qk+a} \equiv m^a \equiv 1 \bmod p$, a contradiction.

Lemma 59 *Any nonzero polynomial of degree k has at most k distinct roots modulo p.*

Exponents and Primitive Roots

- From Fermat's "little" theorem (p. 470), all exponents divide $p-1$.
- A primitive root of p is thus a number with exponent $p-1$.
- Let $R(k)$ denote the total number of residues in $\Phi(p) = \{1, 2, \ldots, p-1\}$ that have exponent k.
- We already knew that $R(k) = 0$ for $k \nmid (p-1)$.
- So
$$\sum_{k \,|\, (p-1)} R(k) = p-1$$
as every number has an exponent.

Size of $R(k)$

- Any $a \in \Phi(p)$ of exponent k satisfies

$$x^k = 1 \bmod p.$$

- By Lemma 59 (p. 475) there are at most k residues of exponent k, i.e., $R(k) \leq k$.
- Let s be a residue of exponent k.
- $1, s, s^2, \ldots, s^{k-1}$ are distinct modulo p.
  - Otherwise, $s^i \equiv s^j \bmod p$ with $i < j$.
  - Then $s^{j-i} = 1 \bmod p$ with $j - i < k$, a contradiction.
- As all these k distinct numbers satisfy $x^k = 1 \bmod p$, they comprise *all* the solutions of $x^k = 1 \bmod p$.

Size of $R(k)$ (continued)

- But do all of them have exponent k (i.e., $R(k) = k$)?
- And if not (i.e., $R(k) < k$), how many of them do?
- Pick s^{ℓ}, where $\ell < k$.
- Suppose $\ell \notin \Phi(k)$ with $\gcd(\ell, k) = d > 1$.
- Then
$$(s^{\ell})^{k/d} = (s^{k})^{\ell/d} = 1 \bmod p.$$
- Therefore, s^{ℓ} has exponent at most $k/d < k$.
- So s^{ℓ} has exponent k *only if* $\ell \in \Phi(k)$.
- We conclude that
$$R(k) \leq \phi(k).$$

Size of $R(k)$ (concluded)

- Because all $p-1$ residues have an exponent,
$$p-1=\sum_{k\,|\,(p-1)} R(k) \leq \sum_{k\,|\,(p-1)} \phi(k)=p-1$$
by Lemma 56 (p. 464).
- Hence
$$R(k)=\begin{cases}\phi(k) & \text{when } k\,|\,(p-1)\\ 0 & \text{otherwise}\end{cases}$$
- In particular, $R(p-1)=\phi(p-1)>0$, and p has at least one primitive root.
- This proves one direction of Theorem 51 (p. 448).

A Few Calculations

- Let $p = 13$.
- From p. 472 $\phi(p-1) = 4$.
- Hence $R(12) = 4$.
- Indeed, there are 4 primitive roots of p.
- As
$$\Phi(p-1) = \{\, 1, 5, 7, 11 \,\},$$
the primitive roots are
$$g^1, g^5, g^7, g^{11},$$
where g is any primitive root.

Function Problems

- Decision problems are yes/no problems (SAT, TSP (D), etc.).
- **Function problems** require a solution (a satisfying truth assignment, a best TSP tour, etc.).
- Optimization problems are clearly function problems.
- What is the relation between function and decision problems?
- Which one is harder?

Function Problems Cannot Be Easier than Decision Problems

- If we know how to generate a solution, we can solve the corresponding decision problem.
  - If you can find a satisfying truth assignment efficiently, then SAT is in P.
  - If you can find the best TSP tour efficiently, then TSP (D) is in P.
- But decision problems can be as hard as the corresponding function problems, as we will see immediately.

FSAT

- FSAT is this function problem:
  - Let $\phi(x_1, x_2, \ldots, x_n)$ be a boolean expression.
  - If ϕ is satisfiable, then return a satisfying truth assignment.
  - Otherwise, return "no."
- We next show that if SAT $\in$ P, then FSAT has a polynomial-time algorithm.
- SAT is a subroutine (black box) that returns "yes" or "no" on the satisfiability of the input.

An Algorithm for FSAT Using SAT

```
1: $t := \epsilon$; {Truth assignment.}
2: if $\phi \in$ SAT then
3:     for $i = 1, 2, \ldots, n$ do
4:         if $\phi[\, x_i = \texttt{true}\,] \in$ SAT then
5:             $t := t \cup \{\, x_i = \texttt{true}\,\}$;
6:             $\phi := \phi[\, x_i = \texttt{true}\,]$;
7:         else
8:             $t := t \cup \{\, x_i = \texttt{false}\,\}$;
9:             $\phi := \phi[\, x_i = \texttt{false}\,]$;
10:        end if
11:    end for
12:    return $t$;
13: else
14:    return "no";
15: end if
```

Analysis

- If SAT can be solved in polynomial time, so can FSAT.
  - There are $\leq n + 1$ calls to the algorithm for SAT.[a]
  - Boolean expressions shorter than ϕ are used in each call to the algorithm for SAT.
- Hence SAT and FSAT are equally hard (or easy).
- Note that this reduction from FSAT to SAT is not a Karp reduction.[b]
- Instead, it calls SAT multiple times as a subroutine, and its answers guide the search on the computation tree.

[a]Contributed by Ms. Eva Ou (`R93922132`) on November 24, 2004.
[b]Recall p. 247 and p. 251.

TSP and TSP (D) Revisited

- We are given n cities $1, 2, \ldots, n$ and integer distances $d_{ij} = d_{ji}$ between any two cities i and j.
- TSP (D) asks if there is a tour with a total distance at most B.
- TSP asks for a tour with the shortest total distance.
  - The shortest total distance is at most $\sum_{i,j} d_{ij}$.
    - Recall that the input string contains $d_{11}, \ldots, d_{nn}$.
- Thus the shortest total distance is less than $2^{|\,x\,|}$ in magnitude, where x is the input (why?).
- We next show that if TSP (D) $\in$ P, then TSP has a polynomial-time algorithm.

An Algorithm for TSP Using TSP (D)

```
1: Perform a binary search over interval $[\,0, 2^{|\,x\,|}\,]$ by calling
   TSP (D) to obtain the shortest distance, $C$;
2: for $i, j = 1, 2, \ldots, n$ do
3:     Call TSP (D) with $B = C$ and $d_{ij} = C + 1$;
4:     if "no" then
5:         Restore $d_{ij}$ to its old value; {Edge $[\,i, j\,]$ is critical.}
6:     end if
7: end for
8: return the tour with edges whose $d_{ij} \leq C$;
```

Analysis

- An edge which is not on *any* remaining optimal tours will be eliminated, with its d_{ij} set to $C + 1$.
- So the algorithm ends with n edges which are not eliminated (why?).
- This is true even if there are multiple optimal tours![a]

[a]Thanks to a lively class discussion on November 12, 2013.

Analysis (concluded)

- There are $O(|\,x\,|+n^2)$ calls to the algorithm for TSP (D).
- Each call has an input length of $O(|\,x\,|)$.
- So if TSP (D) can be solved in polynomial time, so can TSP.
- Hence TSP (D) and TSP are equally hard (or easy).

Randomized Computation

I know that half my advertising works,
I just don't know which half.
— John Wanamaker

I know that half my advertising is
a waste of money,
I just don't know which half!
— McGraw-Hill ad.

Randomized Algorithms[a]

- Randomized algorithms flip unbiased coins.
- There are important problems for which there are no known efficient *deterministic* algorithms but for which very efficient randomized algorithms exist.
  - Extraction of square roots, for instance.
- There are problems where randomization is *necessary*.
  - Secure protocols.
- Randomized version can be more efficient.
  - Parallel algorithms for maximal independent set.[b]

[a]Rabin (1976); Solovay and Strassen (1977).

[b]"Maximal" (a local maximum) not "maximum" (a global maximum).

Randomized Algorithms (concluded)

- Are randomized algorithms algorithms?[a]
- Coin flips are occasionally used in politics.[b]

[a] Pascal, "Truth is so delicate that one has only to depart the least bit from it to fall into error."

[b] In the 2016 Iowa Democratic caucuses, e.g. (see `http://edition.cnn.com/2016/02/02/politics/hillary-clinton-coin-flip-iowa-bernie-sanders/index.html`).

"Four Most Important Randomized Algorithms"[a]

1. Primality testing.[b]
2. Graph connectivity using random walks.[c]
3. Polynomial identity testing.[d]
4. Algorithms for approximate counting.[e]

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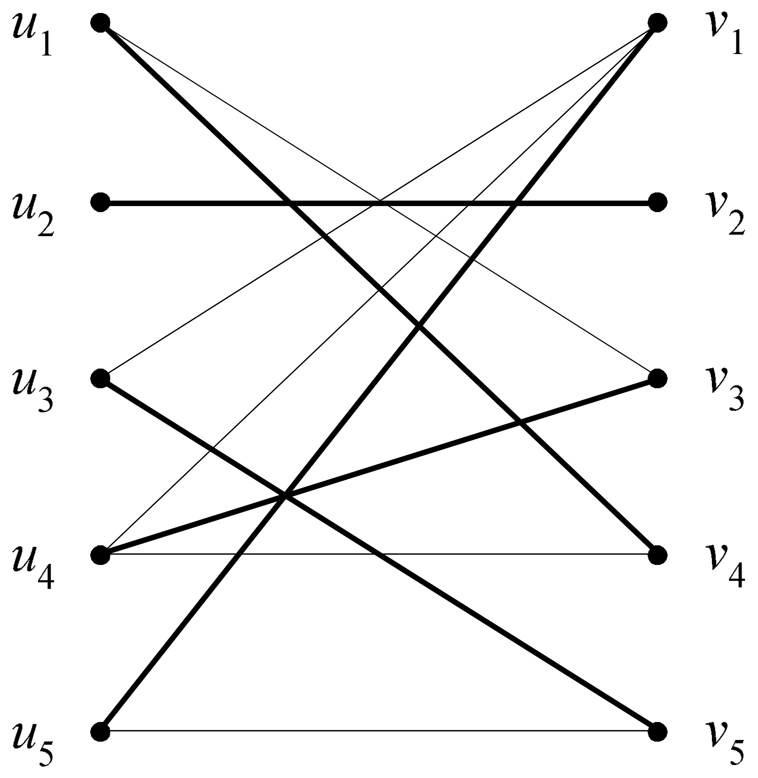
[a]Trevisan (2006).
[b]Rabin (1976); Solovay and Strassen (1977).
[c]Aleliunas, Karp, Lipton, Lovász, and Rackoff (1979).
[d]Schwartz (1980); Zippel (1979).
[e]Sinclair and Jerrum (1989).

Bipartite Perfect Matching

- We are given a **bipartite graph** $G = (U, V, E)$.
  - $U = \{ u_1, u_2, \ldots, u_n \}$.
  - $V = \{ v_1, v_2, \ldots, v_n \}$.
  - $E \subseteq U \times V$.
- We are asked if there is a **perfect matching**.
  - A permutation π of $\{ 1, 2, \ldots, n \}$ such that
  $$(u_i, v_{\pi(i)}) \in E$$
  for all $i \in \{ 1, 2, \ldots, n \}$.
- A perfect matching contains n edges.

A Perfect Matching in a Bipartite Graph



Symbolic Determinants

- We are given a bipartite graph G.
- Construct the $n \times n$ matrix A^G whose (i, j)th entry A^G_{ij} is a symbolic variable x_{ij} if $(u_i, v_j) \in E$ and 0 otherwise:

$$A^G_{ij} = \begin{cases} x_{ij}, & \text{if } (u_i, v_j) \in E, \\ 0, & \text{othersie.} \end{cases}$$

Symbolic Determinants (continued)

- The matrix for the bipartite graph G on p. 496 is[a]

$$A^G = \begin{bmatrix} 0 & 0 & x_{13} & x_{14} & 0 \\ 0 & x_{22} & 0 & 0 & 0 \\ x_{31} & 0 & 0 & 0 & x_{35} \\ x_{41} & 0 & x_{43} & x_{44} & 0 \\ x_{51} & 0 & 0 & 0 & x_{55} \end{bmatrix}. \qquad (6)$$

[a]The idea is similar to the Tanner graph in coding theory by Tanner (1981).

Symbolic Determinants (concluded)

- The **determinant** of A^G is

$$\det(A^G) = \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^{n} A^G_{i,\pi(i)}. \tag{7}$$

  - π ranges over all permutations of n elements.
  - $\operatorname{sgn}(\pi)$ is 1 if π is the product of an even number of transpositions and -1 otherwise.[a]

- $\det(A^G)$ contains $n!$ terms, many of which may be 0s.

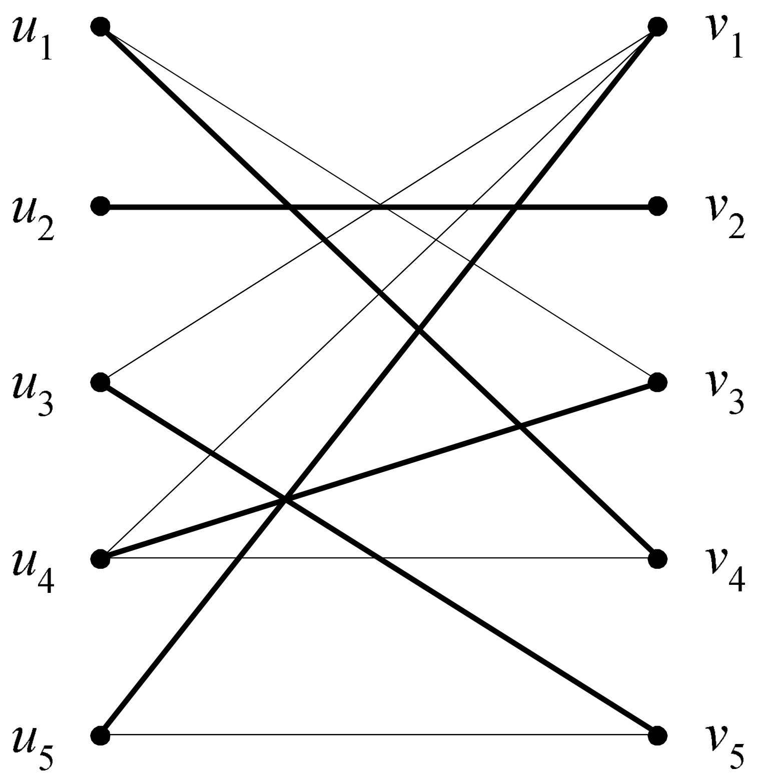
[a]Equivalently, $\operatorname{sgn}(\pi) = 1$ if the number of (i, j)s such that $i < j$ and $\pi(i) > \pi(j)$ is even. Contributed by Mr. Hwan-Jeu Yu (`D95922028`) on May 1, 2008.

Determinant and Bipartite Perfect Matching

- In $\sum_\pi \text{sgn}(\pi) \prod_{i=1}^n A^G_{i,\pi(i)}$, note the following:
  - Each summand corresponds to a possible perfect matching π.
  - Nonzero summands $\prod_{i=1}^n A^G_{i,\pi(i)}$ are distinct monomials and *will not cancel*.
- $\det(A^G)$ is essentially an exhaustive enumeration.

Proposition 60 (Edmonds (1967)) *G has a perfect matching if and only if* $\det(A^G)$ *is not identically zero.*

Perfect Matching and Determinant (p. 496)



Perfect Matching and Determinant (concluded)

- The matrix is (p. 498)

$$A^G = \begin{bmatrix} 0 & 0 & x_{13} & \boxed{x_{14}} & 0 \\ 0 & \boxed{x_{22}} & 0 & 0 & 0 \\ x_{31} & 0 & 0 & 0 & \boxed{x_{35}} \\ x_{41} & 0 & \boxed{x_{43}} & x_{44} & 0 \\ \boxed{x_{51}} & 0 & 0 & 0 & x_{55} \end{bmatrix}.$$

- $\det(A^G) = -x_{14}x_{22}x_{35}x_{43}x_{51} + x_{13}x_{22}x_{35}x_{44}x_{51} + x_{14}x_{22}x_{31}x_{43}x_{55} - x_{13}x_{22}x_{31}x_{44}x_{55}$.
- Each nonzero term denotes a perfect matching, and vice versa.

How To Test If a Polynomial Is Identically Zero?

- $\det(A^G)$ is a polynomial in n^2 variables.
- It has, potentially, exponentially many terms.
- Expanding the determinant polynomial is thus infeasible.
- If $\det(A^G) \equiv 0$, then it remains zero if we substitute *arbitrary* integers for the variables $x_{11}, \ldots, x_{nn}$.
- When $\det(A^G) \not\equiv 0$, what is the likelihood of obtaining a zero?

Number of Roots of a Polynomial

Lemma 61 (Schwartz (1980)) *Let* $p(x_1, x_2, \ldots, x_m) \not\equiv 0$ *be a polynomial in* m *variables each of degree at most* d*. Let* $M \in \mathbb{Z}^+$*. Then the number of* m*-tuples*

$$(x_1, x_2, \ldots, x_m) \in \{0, 1, \ldots, M-1\}^m$$

such that $p(x_1, x_2, \ldots, x_m) = 0$ *is*

$$\leq mdM^{m-1}.$$

- By induction on m (consult the textbook).

Density Attack

- The density of roots in the domain is at most
$$\frac{mdM^{m-1}}{M^m} = \frac{md}{M}. \tag{8}$$
- So suppose $p(x_1, x_2, \ldots, x_m) \not\equiv 0$.
- Then a random
$$(x_1, x_2, \ldots, x_m) \in \{\, 0, 1, \ldots, M-1 \,\}^m$$
has a probability of $\leq md/M$ of being a root of p.
- Note that M is under our control!
  - One can raise M to lower the error probability, e.g.

Density Attack (concluded)

Here is a sampling algorithm to test if $p(x_1, x_2, \ldots, x_m) \not\equiv 0$.

```
1: Choose $i_1, \ldots, i_m$ from $\{ 0, 1, \ldots, M - 1 \}$ randomly;
2: if $p(i_1, i_2, \ldots, i_m) \neq 0$ then
3:     return "$p$ is not identically zero";
4: else
5:     return "$p$ is (probably) identically zero";
6: end if
```

Analysis

- If $p(x_1, x_2, \ldots, x_m) \equiv 0$, the algorithm will always be correct as $p(i_1, i_2, \ldots, i_m) = 0$.
- Suppose $p(x_1, x_2, \ldots, x_m) \not\equiv 0$.
  - The algorithm will answer incorrectly with probability at most md/M by Eq. (8) on p. 505.
- We next return to the original problem of bipartite perfect matching.

A Randomized Bipartite Perfect Matching Algorithm[a]

```
1: Choose $n^2$ integers $i_{11}, \ldots, i_{nn}$ from $\{0, 1, \ldots, 2n^2 - 1\}$
   randomly; {So $M = 2n^2$.}
2: Calculate $\det(A^G(i_{11}, \ldots, i_{nn}))$ by Gaussian elimination;
3: **if** $\det(A^G(i_{11}, \ldots, i_{nn})) \neq 0$ **then**
4:     **return** "$G$ has a perfect matching";
5: **else**
6:     **return** "$G$ has (probably) no perfect matchings";
7: **end if**
```

[a]Lovász (1979). According to Paul Erdős, Lovász wrote his first significant paper "at the ripe old age of 17."

Analysis

- If G has no perfect matchings, the algorithm will always be correct as $\det(A^G(i_{11}, \ldots, i_{nn})) = 0$.
- Suppose G has a perfect matching.
  - The algorithm will answer incorrectly with probability at most $md/M = 0.5$ with $m = n^2$, $d = 1$ and $M = 2n^2$ in Eq. (8) on p. 505.
- Run the algorithm *independently* k times.
- Output "G has no perfect matchings" if and only if *all* say "(probably) no perfect matchings."
- The error probability is now reduced to at most 2^{-k}.

Lószló Lovász (1948–)

Remarks[a]

- Note that we are calculating

$$\text{prob}[\,\text{algorithm answers "no"} \mid G \text{ has no perfect matchings}\,],$$
$$\text{prob}[\,\text{algorithm answers "yes"} \mid G \text{ has a perfect matching}\,].$$

- We are *not* calculating[b]

$$\text{prob}[\,G \text{ has no perfect matchings} \mid \text{algorithm answers "no"}\,],$$
$$\text{prob}[\,G \text{ has a perfect matching} \mid \text{algorithm answers "yes"}\,].$$

[a]Thanks to a lively class discussion on May 1, 2008.

[b]*Numerical Recipes in C* (1988), "statistics is *not* a branch of mathematics!"

But How Large Can $\det(A^G(i_{11}, \ldots, i_{nn}))$ Be?

- It is at most
$$n!\left(2n^2\right)^n.$$
- Stirling's formula says $n! \sim \sqrt{2\pi n}\,(n/e)^n$.
- Hence
$$\log_2 \det(A^G(i_{11}, \ldots, i_{nn})) = O(n \log_2 n)$$
bits are sufficient for representing the determinant.
- We skip the details about how to make sure that all *intermediate* results are of polynomial size.

An Intriguing Question[a]

- Is there an $(i_{11}, \ldots, i_{nn})$ that will always give correct answers for the algorithm on p. 508?
- A theorem on p. 604 shows that such an $(i_{11}, \ldots, i_{nn})$ exists!
  - Whether it can be found efficiently is another matter.
- Once $(i_{11}, \ldots, i_{nn})$ is available, the algorithm can be made deterministic.

[a]Thanks to a lively class discussion on November 24, 2004.

Randomization vs. Nondeterminism[a]

- What are the differences between randomized algorithms and nondeterministic algorithms?
- One can think of a randomized algorithm as a nondeterministic algorithm but with a probability associated with every guess/branch.
- So each computation path of a randomized algorithm has a probability associated with it.

[a]Contributed by Mr. Olivier Valery (`D01922033`) and Mr. Hasan Alhasan (`D01922034`) on November 27, 2012.

Monte Carlo Algorithms[a]

- The randomized bipartite perfect matching algorithm is called a **Monte Carlo algorithm** in the sense that
  - If the algorithm finds that a matching exists, it is always correct (no **false positives**; no **type 1 errors**).
  - If the algorithm answers in the negative, then it may make an error (**false negatives**; **type 2 errors**).

[a]Metropolis and Ulam (1949).

Monte Carlo Algorithms (continued)

- The algorithm makes a false negative with probability ≤ 0.5.[a]
- Again, this probability refers to[b]

  prob[algorithm answers "no" | G has a perfect matching]

  not

  prob[G has a perfect matching | algorithm answers "no"].

[a]Equivalently, among the coin flip sequences, at most half of them lead to the wrong answer.

[b]In general, prob[algorithm answers "no" | input is a yes instance].

Monte Carlo Algorithms (concluded)

- This probability 0.5 is *not* over the space of all graphs or determinants, but *over* the algorithm's own coin flips.
  - It holds for *any* bipartite graph.
- In contrast, to calculate

  $$\text{prob}[\,G \text{ has a perfect matching} \mid \text{algorithm answers "no"}\,],$$

  we will need the distribution of G.
- But it is an empirical statement that is very hard to verify.