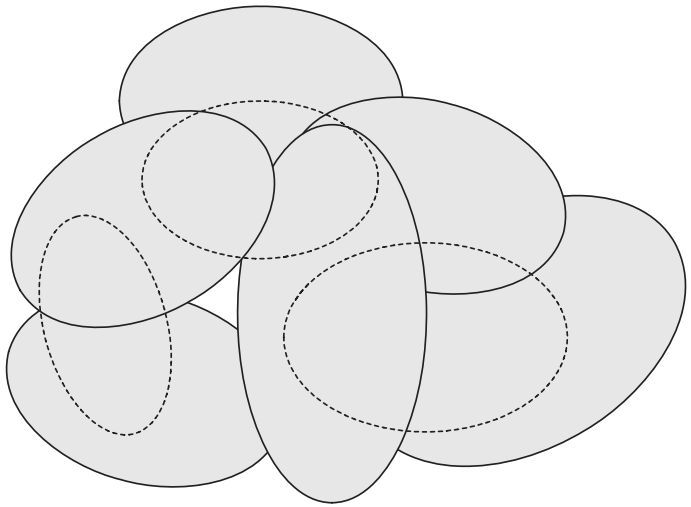
TRIPARTITE MATCHING

- We are given three sets B, G, and H, each containing n elements.
- Let $T \subseteq B \times G \times H$ be a ternary relation.
- TRIPARTITE MATCHING asks if there is a set of n triples in T, none of which has a component in common.
  - Each element in B is matched to a different element in G and different element in H.

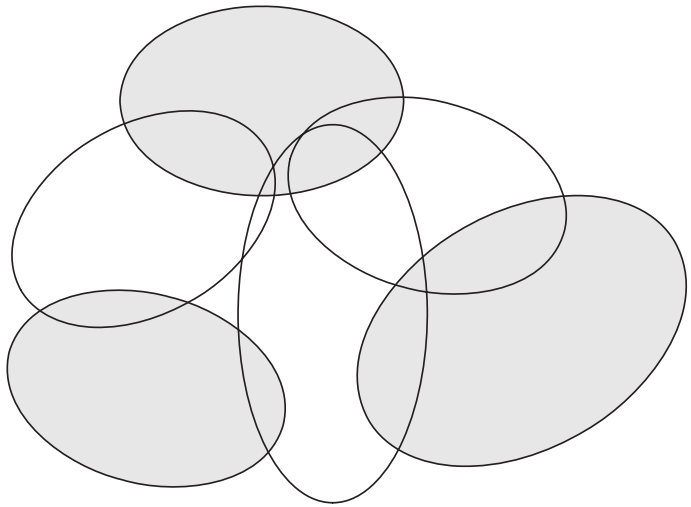
Theorem 45 (Karp (1972)) TRIPARTITE MATCHING *is NP-complete.*

Related Problems

- We are given a family $F = \{ S_1, S_2, \ldots, S_n \}$ of subsets of a finite set U and a budget B.
- SET COVERING asks if there exists a set of B sets in F whose union is U.
- SET PACKING asks if there are B *disjoint* sets in F.
- Assume $| U | = 3m$ for some $m \in \mathbb{N}$ and $| S_i | = 3$ for all i.
- EXACT COVER BY 3-SETS asks if there are m sets in F that are disjoint (so have U as their union).



SET COVERING



SET PACKING

Related Problems (concluded)

Corollary 46 (Karp (1972)) SET COVERING, SET PACKING, *and* EXACT COVER BY 3-SETS *are all NP-complete.*

- Does SET COVERING remain NP-complete when $|S_i| = 3$?[a]
- SET COVERING is used to prove that the influence maximization problem in social networks is NP-complete.[b]

---

[a] Contributed by Mr. Kai-Yuan Hou (B99201038, R03922014) on September 22, 2015.

[b] Kempe, Kleinberg, and Tardos (2003).

KNAPSACK

- There is a set of n items.
- Item i has value $v_i \in \mathbb{Z}^+$ and weight $w_i \in \mathbb{Z}^+$.
- We are given $K \in \mathbb{Z}^+$ and $W \in \mathbb{Z}^+$.
- KNAPSACK asks if there exists a subset

$$I \subseteq \{\, 1, 2, \ldots, n \,\}$$

such that $\sum_{i \in I} w_i \leq W$ and $\sum_{i \in I} v_i \geq K$.

  - We want to achieve the maximum satisfaction within the budget.

KNAPSACK Is NP-Complete[a]

- KNAPSACK $\in$ NP: Guess an I and check the constraints.
- We shall reduce EXACT COVER BY 3-SETS to KNAPSACK, in which $v_i = w_i$ for all i and $K = W$.
- The simplified KNAPSACK now asks if a subset of $v_1, v_2, \ldots, v_n$ adds up to exactly K.[b]
  - Picture yourself as a radio DJ.

[a] Karp (1972).

[b] This problem is called SUBSET SUM or 0-1 KNAPSACK.

The Proof (continued)

- The primary differences between the two problems are:[a]
  - Sets vs. numbers.
  - Union vs. addition.
- We are given a family $F = \{ S_1, S_2, \ldots, S_n \}$ of size-3 subsets of $U = \{ 1, 2, \ldots, 3m \}$.
- EXACT COVER BY 3-SETS asks if there are m disjoint sets in F that cover the set U.

[a] Thanks to a lively class discussion on November 16, 2010.

The Proof (continued)

- Think of a set as a bit vector in $\{ 0, 1 \}^{3m}$.
  - Assume $m = 3$.
  - `110010000` means the set $\{ 1, 2, 5 \}$.
  - `001100010` means the set $\{ 3, 4, 8 \}$.
- Assume there are $n = 5$ size-3 subsets in F.
- Our goal is

$$\overbrace{1\,1\cdots 1}^{3m}.$$

The Proof (continued)

- A bit vector can also be seen as a binary *number*.
- Set union resembles addition:

$$\begin{array}{rr} & \texttt{001100010} \\ + & \texttt{110010000} \\ \hline & \texttt{111110010} \end{array}$$

  which denotes the set $\{ 1, 2, 3, 4, 5, 8 \}$, as desired.

The Proof (continued)

- Trouble occurs when there is *carry*:

$$\begin{array}{rr} & 010000000 \\ + & 010000000 \\ \hline & 100000000 \end{array}$$

  which denotes the wrong set $\{\,1\,\}$, not the correct $\{\,2\,\}$.

The Proof (continued)

- Or consider

$$\begin{array}{rr} & \texttt{001100010} \\ + & \texttt{001110000} \\ \hline & \texttt{011010010} \end{array}$$

  which denotes the set $\{ 2, 3, 5, 8 \}$, not the correct $\{ 3, 4, 5, 8 \}$.[a]

[a] Corrected by Mr. Chihwei Lin (D97922003) on January 21, 2010.

The Proof (continued)

- Carry may also lead to a situation where we obtain our solution $\mathtt{1\,1}\cdots\mathtt{1}$ with more than m sets in F.
- For example, with $m = 3$,

$$\begin{array}{rr} & \mathtt{000100010} \\ & \mathtt{001110000} \\ & \mathtt{101100000} \\ + & \mathtt{000001101} \\ \hline & \mathtt{111111111} \end{array}$$

- But the correct union result, $\{\,1, 3, 4, 5, 6, 7, 8, 9\,\}$, is *not* an exact cover.

The Proof (continued)

- And it uses 4 sets instead of the required $m = 3$.[a]
- To fix this problem, we enlarge the base just enough so that there are no carries.[b]
- Because there are n vectors in total, we change the base from 2 to $n + 1$.
- Every positive integer N has a unique expression in base b: There are **b-adic digits** $0 \leq d_i < b$ such that

$$N = \sum_{i=0}^{k} d_i b^i, \quad d_k \neq 0.$$

[a] Thanks to a lively class discussion on November 20, 2002.

[b] You cannot map $\cup$ to $\vee$ because KNAPSACK requires $+$ not $\vee$!

The Proof (continued)

- Set v_i to be the integer corresponding to the bit vector encoding S_i in base $n+1$:

$$v_i = \sum_{j \in S_i} 1 \times (n+1)^{3m-j} \tag{3}$$

- Set

$$K = \sum_{j=0}^{3m-1} 1 \times (n+1)^j = \overbrace{1\,1 \cdots 1}^{3m} \quad (\text{base } n+1).$$

- Now in base $n+1$, if there is a set S such that $\sum_{i \in S} v_i = \overbrace{1\,1 \cdots 1}^{3m}$, then every position must be contributed by exactly one v_i and $|\,S\,| = m$.

The Proof (continued)

- For example, the case on p. 413 becomes

$$\begin{array}{rr} & 000100010 \\ & 001110000 \\ & 101100000 \\ + & 000001101 \\ \hline & 102311111 \end{array}$$

  in base $n + 1 = 6$.

- As desired, it no longer meets the goal.

The Proof (continued)

- Suppose F admits an exact cover, say $\{ S_1, S_2, \ldots, S_m \}$.
- Then picking $I = \{ 1, 2, \ldots, m \}$ clearly results in

$$v_1 + v_2 + \cdots + v_m = \overbrace{11\cdots1}^{3m}.$$

- It is important to note that the meaning of addition (+) is independent of the base.[a]
  - It is just regular addition.
  - But an S_i may give rise to different integers v_i in Eq. (3) on p. 415 under different bases.

[a]Contributed by Mr. Kuan-Yu Chen (R92922047) on November 3, 2004.

The Proof (concluded)

- On the other hand, suppose there exists an I such that

$$\sum_{i \in I} v_i = \overbrace{1\,1 \cdots 1}^{3m}$$

  in base $n+1$.

- The no-carry property implies that $|\, I \,| = m$ and

$$\{\, S_i : i \in I \,\}$$

  is an exact cover.

The proof actually proves:

Corollary 47 SUBSET SUM *(p. 407) is NP-complete.*

An Example

- Let $m = 3$, $U = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9 \}$, and

$$\begin{aligned} S_1 &= \{ 1, 3, 4 \}, \\ S_2 &= \{ 2, 3, 4 \}, \\ S_3 &= \{ 2, 5, 6 \}, \\ S_4 &= \{ 6, 7, 8 \}, \\ S_5 &= \{ 7, 8, 9 \}. \end{aligned}$$

- Note that $n = 5$, as there are 5 S_i's.

An Example (continued)

- Our reduction produces

$$
\begin{aligned}
K &= \sum_{j=0}^{3\times 3-1} 6^j = \overbrace{11\cdots 1}^{3\times 3}{}_6 = 2015539_{10}, \\
v_1 &= \texttt{101100000} = 1734048, \\
v_2 &= \texttt{011100000} = 334368, \\
v_3 &= \texttt{010011000} = 281448, \\
v_4 &= \texttt{000001110} = 258, \\
v_5 &= \texttt{000000111} = 43.
\end{aligned}
$$

An Example (concluded)

- Note $v_1 + v_3 + v_5 = K$ because

$$\begin{array}{rr} & \texttt{101100000} \\ & \texttt{010011000} \\ + & \texttt{000000111} \\ \hline & \texttt{111111111} \end{array}$$

- Indeed,

$$S_1 \cup S_3 \cup S_5 = \{\, 1, 2, 3, 4, 5, 6, 7, 8, 9 \,\},$$

  an exact cover by 3-sets.

BIN PACKING

- We are given N positive integers $a_1, a_2, \ldots, a_N$, an integer C (the capacity), and an integer B (the number of bins).
- BIN PACKING asks if these numbers can be partitioned into B subsets, each of which has total sum at most C.
- Think of packing bags at the check-out counter.

Theorem 48 BIN PACKING *is NP-complete.*

BIN PACKING (concluded)

- But suppose $a_1, a_2, \ldots, a_N$ are randomly distributed between 0 and 1.
- Let B be the smallest number of unit-capacity bins capable of holding them.
- Then B can deviate from its average by more than t with probability at most $2e^{-2t^2/N}$.[a]

[a] Dubhashi and Panconesi (2012).

INTEGER PROGRAMMING (IP)

- IP asks whether a system of linear inequalities with integer coefficients has an integer solution.
- In contrast, LINEAR PROGRAMMING (LP) asks whether a system of linear inequalities with integer coefficients has a *rational* solution.
  - LP is solvable in polynomial time.[a]

[a] Khachiyan (1979).

IP Is NP-Complete[a]

- SET COVERING can be expressed by the inequalities $Ax \geq \vec{1}$, $\sum_{i=1}^{n} x_i \leq B$, $0 \leq x_i \leq 1$, where
  - $x_i = 1$ if and only if S_i is in the cover.
  - A is the matrix whose columns are the bit vectors of the sets $S_1, S_2, \ldots$.
  - $\vec{1}$ is the vector of 1s.
  - The operations in Ax are standard matrix operations.
  - The ith row of Ax is at least 1 means item i is covered.

[a]Karp (1972); Papadimitriou (1981).

IP Is NP-Complete (concluded)

- This shows IP is NP-hard.
- Many NP-complete problems can be expressed as an IP problem.

Christos Papadimitriou (1949–)

Easier or Harder?[a]

- Adding restrictions on the allowable *problem instances* will not make a problem harder.
  - We are now solving a subset of problem instances or special cases.
  - The INDEPENDENT SET proof (p. 350) and the KNAPSACK proof (p. 407): equally hard.
  - CIRCUIT VALUE to MONOTONE CIRCUIT VALUE (p. 299): equally hard.
  - SAT to 2SAT (p. 330): easier.

[a]Thanks to a lively class discussion on October 29, 2003.

Easier or Harder? (concluded)

- Adding restrictions on the allowable *solutions* (the solution space) may make a problem harder, equally hard, or easier.
- It is problem dependent.
  - MIN CUT to BISECTION WIDTH (p. 382): harder.
  - LP to IP (p. 424): harder.
  - SAT to NAESAT (equally hard by p. 342) and MAX CUT to MAX BISECTION (p. 380): equally hard.
  - 3-COLORING to 2-COLORING (p. 391): easier.

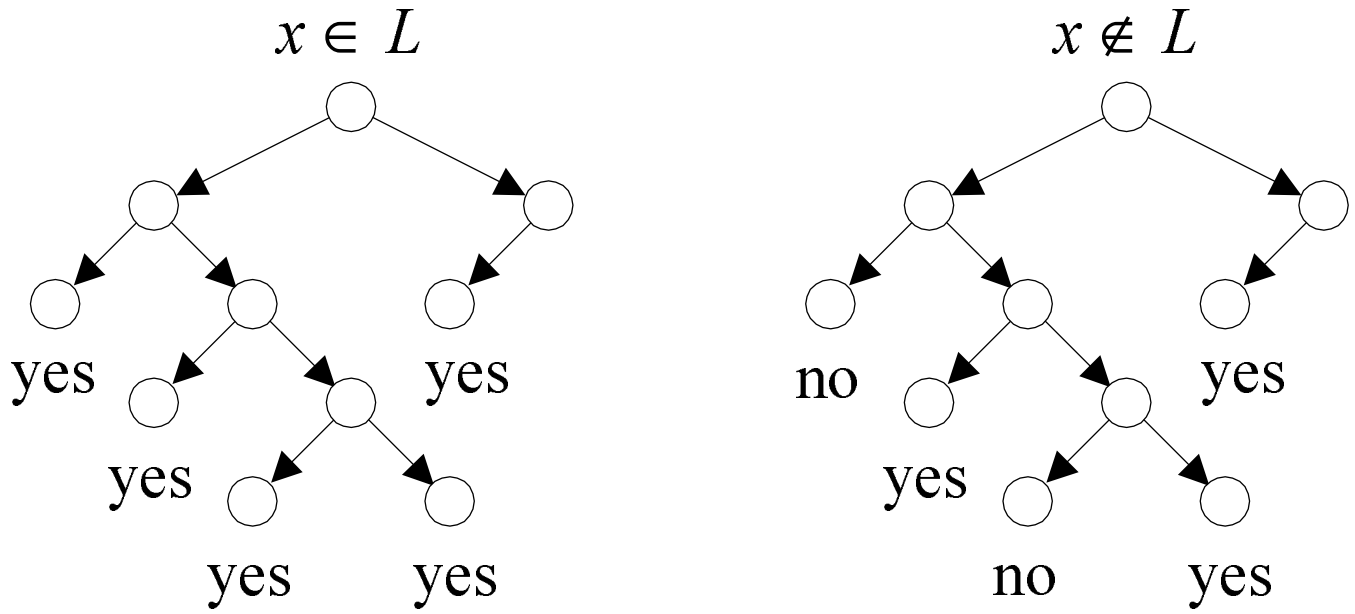
coNP and Function Problems

coNP

- NP is the class of problems that have succinct certificates (recall Proposition 36 on p. 312).
- By definition, coNP is the class of problems whose complement is in NP.
- coNP is therefore the class of problems that have succinct disqualifications:
  - A "no" instance of a problem in coNP possesses a short proof of its being a "no" instance.
  - Only "no" instances have such proofs.

coNP (continued)

- Suppose L is a coNP problem.
- There exists a polynomial-time nondeterministic algorithm M such that:
  - If $x \in L$, then $M(x) =$ "yes" for all computation paths.
  - If $x \notin L$, then $M(x) =$ "no" for some computation path.
- Note that if we swap "yes" and "no" of M, the new algorithm M' decides $\bar{L} \in$ NP in the classic sense (p. 104).


$x \in L$
yes
yes
yes
yes
yes
$x \notin L$
no
yes
yes
no
yes

coNP (continued)

- So there are 3 major approaches to proving $L \in \text{coNP}$.
  1. Prove $\bar{L} \in \text{NP}$.
  2. Prove that only "no" instances possess short proofs.
  3. Write an algorithm for it directly.

coNP (concluded)

- Clearly $\mathrm{P} \subseteq \mathrm{coNP}$.
- It is not known if

$$\mathrm{P} = \mathrm{NP} \cap \mathrm{coNP}.$$

  - Contrast this with

$$\mathrm{R} = \mathrm{RE} \cap \mathrm{coRE}$$

    (see Proposition 10 on p. 150).

Some coNP Problems

- VALIDITY $\in$ coNP.
  - If ϕ is not valid, it can be disqualified very succinctly: a truth assignment that does not satisfy it.
- SAT COMPLEMENT $\in$ coNP.
  - SAT COMPLEMENT is the complement of SAT.
  - The disqualification is a truth assignment that satisfies it.
- HAMILTONIAN PATH COMPLEMENT $\in$ coNP.
  - The disqualification is a Hamiltonian path.

Some coNP Problems (concluded)

- OPTIMAL TSP (D) $\in$ coNP.
  - OPTIMAL TSP (D) asks if the optimal tour has a total distance of B, where B is an input.[a]
  - The disqualification is a tour with a length $< B$.

[a] Defined by Mr. Che-Wei Chang (R95922093) on September 27, 2006.

A Nondeterministic Algorithm for SAT COMPLEMENT (See also p. 114)

ϕ is a boolean formula with n variables.

1: **for** $i = 1, 2, \ldots, n$ **do**
2: Guess $x_i \in \{ 0, 1 \}$; {Nondeterministic choice.}
3: **end for**
4: {Verification:}
5: **if** $\phi(x_1, x_2, \ldots, x_n) = 1$ **then**
6: "no";
7: **else**
8: "yes";
9: **end if**

Analysis

- The algorithm decides language $\{\,\phi : \phi \text{ is unsatisfiable}\,\}$.
  - The computation tree is a complete binary tree of depth n.
  - Every computation path corresponds to a particular truth assignment out of 2^n.
  - ϕ is unsatisfiable if and only if every truth assignment falsifies ϕ.
  - But every truth assignment falsifies ϕ if and only if every computation path results in "yes."

An Alternative Characterization of coNP

Proposition 49 *Let $L \subseteq \Sigma^*$ be a language. Then $L \in coNP$ if and only if there is a polynomially decidable and polynomially balanced relation R such that*

$$L = \{\, x : \forall y\, (x, y) \in R \,\}.$$

(As on p. 311, we assume $|\, y \,| \leq |\, x \,|^k$ for some k.)

- $\bar{L} = \{\, x : \exists y\, (x, y) \in \neg R \,\}$.
- Because $\neg R$ remains polynomially balanced, $\bar{L} \in$ NP by Proposition 36 (p. 312).
- Hence $L \in$ coNP by definition.

coNP-Completeness

Proposition 50 *L is NP-complete if and only if its complement $\bar{L} = \Sigma^* - L$ is coNP-complete.*

Proof ($\Rightarrow$; the $\Leftarrow$ part is symmetric)

- Let $\bar{L}'$ be any coNP language.
- Hence $L' \in$ NP.
- Let R be the reduction from L' to L.
- So $x \in L'$ if and only if $R(x) \in L$.
- By the law of transposition, $x \notin L'$ if and only if $R(x) \notin L$.

coNP Completeness (concluded)

- So $x \in \bar{L}'$ if and only if $R(x) \in \bar{L}$.
- The *same* R is a reduction from $\bar{L}'$ to $\bar{L}$.
- This shows $\bar{L}$ is coNP-hard.
- But $\bar{L} \in$ coNP.
- This shows $\bar{L}$ is coNP-complete.

Some coNP-Complete Problems

- SAT COMPLEMENT is coNP-complete.
- VALIDITY is coNP-complete.
  - ϕ is valid if and only if $\neg\phi$ is not satisfiable.
  - $\phi \in$ VALIDITY is valid if and only if $\neg\phi \in$ SAT COMPLEMENT.
  - The reduction from SAT COMPLEMENT to VALIDITY is hence easy.
- HAMILTONIAN PATH COMPLEMENT is coNP-complete.

Possible Relations between P, NP, coNP

1. $\mathrm{P} = \mathrm{NP} = \mathrm{coNP}$.
2. $\mathrm{NP} = \mathrm{coNP}$ but $\mathrm{P} \neq \mathrm{NP}$.
3. $\mathrm{NP} \neq \mathrm{coNP}$ and $\mathrm{P} \neq \mathrm{NP}$.
   - This is the current "consensus."[a]

[a] Carl Gauss (1777–1855), "I could easily lay down a multitude of such propositions, which one could neither prove nor dispose of."

The Primality Problem

- An integer p is **prime** if $p > 1$ and all positive numbers other than 1 and p itself cannot divide it.
- PRIMES asks if an integer N is a prime number.
- Dividing N by $2, 3, \ldots, \sqrt{N}$ is *not* efficient.
  - The length of N is only $\log N$, but $\sqrt{N} = 2^{0.5 \log N}$.
  - It is an exponential-time algorithm.
- A polynomial-time algorithm for PRIMES was not found until 2002 by Agrawal, Kayal, and Saxena!
- The running time is $\tilde{O}(\log^{7.5} N)$.

```
 1: if n = a^b for some a, b > 1 then
 2:    return "composite";
 3: end if
 4: for r = 2, 3, ..., n − 1 do
 5:    if gcd(n, r) > 1 then
 6:       return "composite";
 7:    end if
 8:    if r is a prime then
 9:       Let q be the largest prime factor of r − 1;
10:       if q ≥ 4√r log n and n^((r−1)/q) ≠ 1 mod r then
11:          break; {Exit the for-loop.}
12:       end if
13:    end if
14: end for{r − 1 has a prime factor q ≥ 4√r log n.}
15: for a = 1, 2, ..., 2√r log n do
16:    if (x − a)^n ≠ (x^n − a) mod (x^r − 1) in Z_n[ x ] then
17:       return "composite";
18:    end if
19: end for
20: return "prime"; {The only place with "prime" output.}
```

The Primality Problem (concluded)

- Later, we will focus on efficient "randomized" algorithms for PRIMES (used in *Mathematica*, e.g.).
- NP $\cap$ coNP is the class of problems that have succinct certificates and succinct disqualifications.
  - Each "yes" instance has a succinct certificate.
  - Each "no" instance has a succinct disqualification.
  - No instances have both.
- We will see that PRIMES $\in$ NP $\cap$ coNP.[a]
  - In fact, PRIMES $\in$ P as mentioned earlier.

[a] Another important problem in that class is LP.

Primitive Roots in Finite Fields

Theorem 51 (Lucas and Lehmer (1927)) [a] *A number $p > 1$ is a prime if and only if there is a number $1 < r < p$ such that*

1. $r^{p-1} = 1 \bmod p$, *and*
2. $r^{(p-1)/q} \neq 1 \bmod p$ *for all prime divisors q of $p - 1$.*

- This r is called the **primitive root** or **generator**.
- We will prove one direction of the theorem later.[b]

---

[a] François Edouard Anatole Lucas (1842–1891); Derrick Henry Lehmer (1905–1991).

[b] See pp. 461ff.

Derrick Lehmer[a] (1905–1991)

[a]Inventor of the linear congruential generator in 1951.

Pratt's Theorem

Theorem 52 (Pratt (1975)) PRIMES $\in NP \cap coNP$.

- PRIMES is in coNP because a succinct disqualification is a proper divisor.
  - A proper divisor of a number n means n is *not* a prime.
- Now suppose p is a prime.
- p's certificate includes the r in Theorem 51 (p. 448).
  - There may be multiple choices for r.

The Proof (continued)

- Use recursive doubling to check if $r^{p-1} = 1 \bmod p$ in time polynomial in the length of the input, $\log_2 p$.
  - $r, r^2, r^4, \ldots \bmod p$, a total of $\sim \log_2 p$ steps.
- We also need all *prime* divisors of $p-1$: $q_1, q_2, \ldots, q_k$.
  - Whether $r, q_1, \ldots, q_k$ are easy to find is irrelevant.
- Checking $r^{(p-1)/q_i} \neq 1 \bmod p$ is also easy.
- Checking $q_1, q_2, \ldots, q_k$ are all the divisors of $p-1$ is easy.

The Proof (concluded)

- We still need certificates for the primality of the q_i's.
- The complete certificate is recursive and tree-like:

$$C(p) = (r; q_1, C(q_1), q_2, C(q_2), \ldots, q_k, C(q_k)). \quad (4)$$

- We next prove that $C(p)$ is succinct.
- As a result, $C(p)$ can be checked in polynomial time.

The Succinctness of the Certificate

Lemma 53 *The length of $C(p)$ is at most quadratic at $5 \log_2^2 p$.*

- This claim holds when $p = 2$ or $p = 3$.
- In general, $p - 1$ has $k \leq \log_2 p$ prime divisors $q_1 = 2, q_2, \ldots, q_k$.
  - Reason:

$$2^k \leq \prod_{i=1}^{k} q_i \leq p - 1.$$

- Note also that, as $q_1 = 2$,

$$\prod_{i=2}^{k} q_i \leq \frac{p-1}{2}. \tag{5}$$

The Proof (continued)

- $C(p)$ requires:
  - 2 parentheses;
  - $2k < 2\log_2 p$ separators (at most $2\log_2 p$ bits);
  - r (at most $\log_2 p$ bits);
  - $q_1 = 2$ and its certificate 1 (at most 5 bits);
  - $q_2, \ldots, q_k$ (at most $2\log_2 p$ bits);[a]
  - $C(q_2), \ldots, C(q_k)$.

[a] Why?

The Proof (concluded)

- $C(p)$ is succinct because, by induction,

$$
\begin{aligned}
|\, C(p) \,| &\leq 5 \log_2 p + 5 + 5 \sum_{i=2}^{k} \log_2^2 q_i \\
&\leq 5 \log_2 p + 5 + 5 \left(\sum_{i=2}^{k} \log_2 q_i \right)^2 \\
&\leq 5 \log_2 p + 5 + 5 \log_2^2 \frac{p-1}{2} \quad \text{by inequality (5)} \\
&< 5 \log_2 p + 5 + 5[\, (\log_2 p) - 1 \,]^2 \\
&= 5 \log_2^2 p + 10 - 5 \log_2 p \leq 5 \log_2^2 p
\end{aligned}
$$

for $p \geq 4$.

A Certificate for 23[a]

- Note that 5 is a primitive root modulo 23 and $23 - 1 = 22 = 2 \times 11$.[b]
- So
$$C(23) = (5; 2, C(2), 11, C(11)).$$
- Note that 2 is a primitive root modulo 11 and $11 - 1 = 10 = 2 \times 5$.
- So
$$C(11) = (2; 2, C(2), 5, C(5)).$$

[a] Thanks to a lively discussion on April 24, 2008.

[b] Other primitive roots are $7, 10, 11, 14, 15, 17, 19, 20, 21$.

A Certificate for 23 (concluded)

- Note that 2 is a primitive root modulo 5 and $5 - 1 = 4 = 2^2$.
- So

$$C(5) = (2; 2, C(2)).$$

- In summary,

$$C(23) = (5; 2, C(2), 11, (2; 2, C(2), 5, (2; 2, C(2)))).$$

  - In *Mathematica*, `PrimeQCertificate[23]` yields

$$\{\, 23, 5, \{\, 2, \{\, 11, 2, \{\, 2, \{\, 5, 2, \{\, 2 \,\}\}\}\}\}\}$$

Turning the Proof into an Algorithm[a]

- How to turn the proof into a polynomial-time nondeterministic algorithm?
- First, guess a $\log_2 p$-bit number r.
- Then guess up to $\log_2 p$ numbers $q_1, q_2, \ldots, q_k$ each containing at most $\log_2 p$ bits.
- Then recursively do the same thing for each of the q_i to form a certificate (4) on p. 452.
- Finally check if the two conditions of Theorem 51 (p. 448) hold throughout the tree.

[a]Contributed by Mr. Kai-Yuan Hou (B99201038, R03922014) on November 24, 2015.