Nondeterministic Space Complexity Classes

- Let L be a language.
- Then
$$L \in \text{NSPACE}(f(n))$$
if there is an NTM with input and output that decides L and operates within space bound $f(n)$.
- $\text{NSPACE}(f(n))$ is a set of languages.
- As in the linear speedup theorem,[a], constant coefficients do not matter.

[a] Theorem 5 (p. 90).

Graph Reachability

- Let $G(V, E)$ be a directed graph (**digraph**).
- REACHABILITY asks, given nodes a and b, does G contain a path from a to b?
- Can be easily solved in polynomial time by breadth-first search.
- How about its nondeterministic space complexity?

The First Try: NSPACE$(n \log n)$

Determine the number of nodes m; {Note $m \leq n$.}
$x_1 := a$; {Assume $a \neq b$.}
for $i = 2, 3, \ldots, m$ **do**
    Guess $x_i \in \{ v_1, v_2, \ldots, v_m \}$; {The ith node.}
end for
for $i = 2, 3, \ldots, m$ **do**
    if $(x_{i-1}, x_i) \notin E$ **then**
        "no";
    end if
    if $x_i = b$ **then**
        "yes";
    end if
end for
"no";

In Fact, REACHABILITY $\in$ NSPACE$(\log n)$

```
Determine the number of nodes m; {Note m ≤ n.}
x := a;
for i = 2, 3, ..., m do
    Guess y ∈ { v_1, v_2, ..., v_m }; {The next node.}
    if (x, y) ∉ E then
        "no";
    end if
    if y = b then
        "yes";
    end if
    x := y;
end for
"no";
```

Space Analysis

- Variables m, i, x, and y each require $O(\log n)$ bits.
- Testing $(x, y) \in E$ is accomplished by consulting the input string with counters of $O(\log n)$ bits long.
- Hence
$$\text{REACHABILITY} \in \text{NSPACE}(\log n).$$
  - REACHABILITY with more than one terminal node also has the same complexity.
- REACHABILITY $\in$ P (see, e.g., p. 223).

Undecidability

He [Turing] invented
the idea of software, essentially[.]
It's software that's really
the important invention.
— Freeman Dyson (2015)

Universal Turing Machine[a]

- A **universal Turing machine** U interprets the input as the *description* of a TM M concatenated with the *description* of an input to that machine, x.
  - Both M and x are over the alphabet of U.
- U simulates M on x so that

$$U(M; x) = M(x).$$

- U is like a modern computer, which executes any valid machine code, or a Java virtual machine, which executes any valid bytecode.

[a]Turing (1936).

The Halting Problem

- **Undecidable problems** are problems that have no algorithms.
  - Equivalently, they are languages that are not recursive.
- We now define a concrete undecidable problem, the **halting problem**:

$$H = \{ M; x : M(x) \neq \nearrow \}.$$

  - Does M halt on input x?

H Is Recursively Enumerable

- Use the universal TM U to simulate M on x.
- When M is about to halt, U enters a "yes" state.
- If $M(x)$ diverges, so does U.
- This TM accepts H.

H Is Not Recursive[a]

- Suppose H is recursive.
- Then there is a TM M_H that *decides* H.
- Consider the program $D(M)$ that calls M_H:

1: **if** $M_H(M; M) =$ "yes" **then**
2: $\quad\nearrow$; {Writing an infinite loop is easy.}
3: **else**
4: $\quad$"yes";
5: **end if**

---

[a] Turing (1936).

H Is Not Recursive (concluded)

- Consider $D(D)$:
  - $D(D) = \nearrow \Rightarrow M_H(D; D) =$ "yes" $\Rightarrow D; D \in H \Rightarrow D(D) \neq \nearrow$, a contradiction.
  - $D(D) =$ "yes" $\Rightarrow M_H(D; D) =$ "no" $\Rightarrow D; D \notin H \Rightarrow D(D) = \nearrow$, a contradiction.

Comments

- Two levels of interpretations of M:[a]
  - A sequence of 0s and 1s (data).
  - An encoding of instructions (programs).
- There are no paradoxes with $D(D)$.
  - Concepts should be familiar to computer scientists.
  - Feed a C compiler to a C compiler, a Lisp interpreter to a Lisp interpreter, a sorting program to a sorting program, etc.

---

[a] Eckert and Mauchly (1943); von Neumann (1945); Turing (1946).

It seemed unworthy of a grown man
to spend his time on such trivialities,
but what was I to do? $[\cdots]$
The whole of the rest of my life might be
consumed in looking at
that blank sheet of paper.
— Bertrand Russell (1872–1970),
Autobiography, Vol. I (1967)

Self-Loop Paradoxes[a]

Russell's Paradox (1901): Consider $R = \{A : A \notin A\}$.

- If $R \in R$, then $R \notin R$ by definition.
- If $R \notin R$, then $R \in R$ also by definition.
- In either case, we have a "contradiction."[b]

Eubulides: The Cretan says, "All Cretans are liars."

Liar's Paradox: "This sentence is false."

---

[a] E.g., Quine (1966), *The Ways of Paradox and Other Essays* and Hofstadter (1979), *Gödel, Escher, Bach: An Eternal Golden Braid*.

[b] Gottlob Frege (1848–1925) to Bertrand Russell in 1902, "Your discovery of the contradiction [...] has shaken the basis on which I intended to build arithmetic."

Self-Loop Paradoxes (continued)

Hypochondriac: a patient (like Gödel) with imaginary symptoms and ailments.

Sharon Stone in *The Specialist* (1994): "I'm not a woman you can trust."

***Numbers* 12:3, Old Testament:** "Moses was the most humble person in all the world $[\cdots]$" (attributed to Moses).

Soren Kierkegaard in *Fear and Trembling* (1843): "to strive against the whole world is a comfort, to strive with oneself is dreadful."

Self-Loop Paradoxes (concluded)

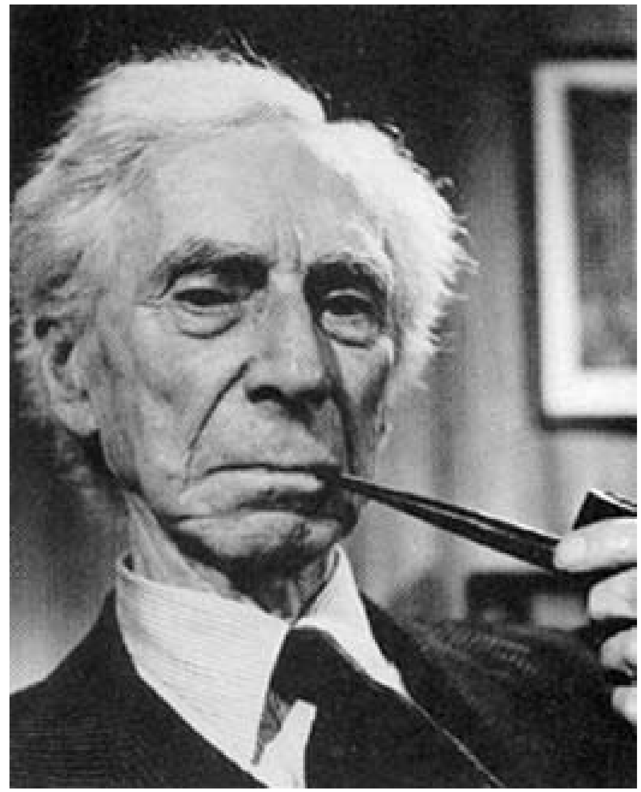
The Egyptian Book of the Dead**:** "ye live in me and I would live in you."

***John* 14:10, New Testament:** "Don't you believe that I am in the Father, and that the Father is in me?"

***John* 17:21, New Testament:** "just as you are in me and I am in you."

***Pagan & Christian Creeds* (1920):** "I was moved to Odin, myself to myself."

Bertrand Russell[a] (1872–1970)



Karl Popper (1974), "perhaps the greatest philosopher since Kant."

[a]Nobel Prize in Literature (1950).

Reductions in Proving Undecidability

- Suppose we are asked to prove that L is undecidable.
- Suppose L' (such as H) is known to be undecidable.
- Find a computable transformation R (called **reduction**[a]) from L' to L such that[b]

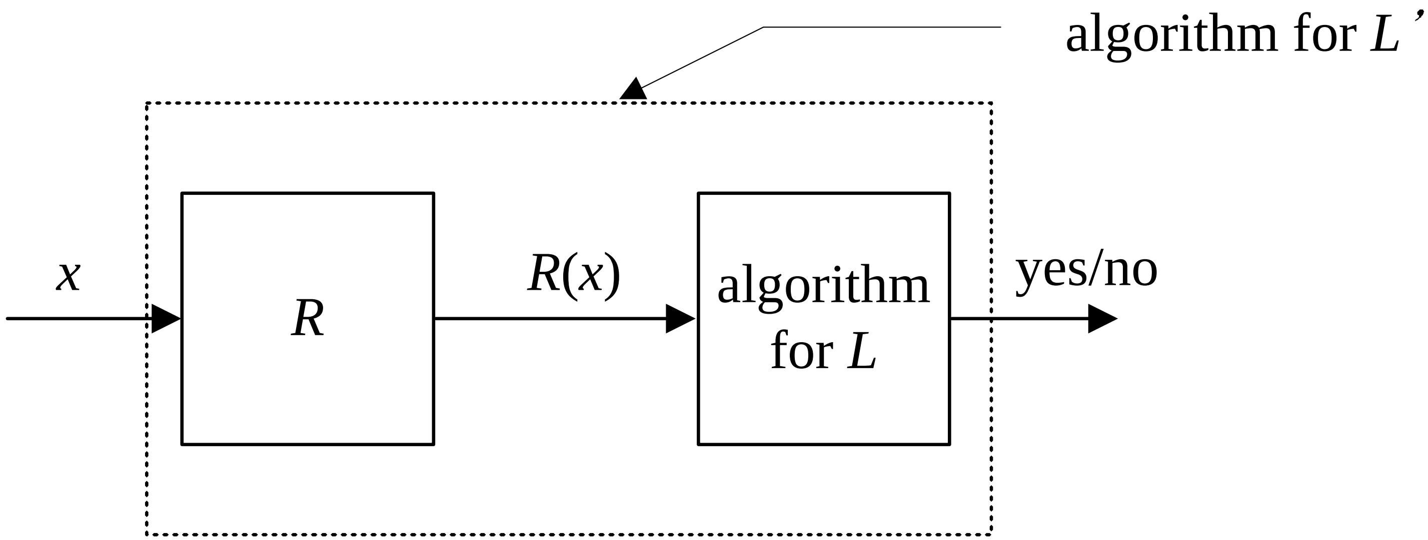
$$\forall x \, \{ \, x \in L' \text{ if and only if } R(x) \in L \, \}.$$

- Now we can answer "$x \in L'$?" for *any* x by answering "$R(x) \in L$?" because it has the same answer.
- L' is said to be **reduced** to L.

---

[a] Post (1944).

[b] Contributed by Mr. Tai-Dai Chou (J93922005) on May 19, 2005.


algorithm for L'
x
R
R(x)
algorithm for L
yes/no

Reductions in Proving Undecidability (concluded)

- If L were decidable, "$R(x) \in L$?" becomes computable and we have an algorithm to decide L', a contradiction!
- So L must be undecidable.

Theorem 8 *Suppose language L_1 can be reduced to language L_2. If L_1 is undecidable, then L_2 is undecidable.*

Special Cases and Reduction

- Suppose L_1 can be reduced to L_2.
- As the reduction R maps members of L_1 to a *subset* of L_2,[a] we may say L_1 is a "special case" of L_2.[b]
- That is one way to understand the use of the term "reduction."

---

[a] Because R may not be onto.

[b] Contributed by Ms. Mei-Chih Chang (`D03922022`) and Mr. Kai-Yuan Hou (`B99201038`, `R03922014`) on October 13, 2015.

Subsets and Decidability

- Suppose L_1 is undecidable and $L_1 \subseteq L_2$.
- Is L_2 undecidable?[a]
- It depends.
- When $L_2 = \Sigma^*$, L_2 is decidable: Just answer "yes."
- If $L_2 - L_1$ is decidable, then L_2 is undecidable.
  - Clearly,

$$x \in L_1 \text{ if and only if } x \in L_2 \text{ and } x \notin L_2 - L_1.$$

  - Therefore, if L_2 were decidable, then L_1 would be.

[a]Contributed by Ms. Mei-Chih Chang (D03922022) on October 13, 2015.

The Universal Halting Problem

- The **universal halting problem**:

$$H^* = \{ M : M \text{ halts on all inputs} \}.$$

- It is also called **the totality problem**.

H^* Is Not Recursive[a]

- We will reduce H to H^*.
- Given the question "$M; x \in H$?", construct the following machine (this is the reduction):[b]

$$M_x(y)\ \{M(x);\}$$

- M halts on x if and only if M_x halts on all inputs.
- In other words, $M; x \in H$ if and only if $M_x \in H^*$.
- So if H^* were recursive (recall the box for L on p. 142), H would be recursive, a contradiction.

---

[a]Kleene (1936).

[b]Simplified by Mr. Chih-Hung Hsieh (`D95922003`) on October 5, 2006. M_x ignores its input y; x is part of M_x's code but not M_x's input.

More Undecidability

- $\{ M; x : \text{there is a } y \text{ such that } M(x) = y \}$.
- $\{ M; x :$
  the computation M on input x uses all states of $M \}$.
- $L = \{ M; x; y : M(x) = y \}$.

Complements of Recursive Languages

The **complement** of L, denoted by $\bar{L}$, is the language $\Sigma^* - L$.

Lemma 9 *If L is recursive, then so is $\bar{L}$.*

- Let L be decided by M, which is deterministic.
- Swap the "yes" state and the "no" state of M.
- The new machine decides $\bar{L}$.[a]

---

[a]Recall p. 106.

Recursive and Recursively Enumerable Languages

Lemma 10 (Kleene's theorem; Post (1944)) *L is recursive if and only if both L and $\bar{L}$ are recursively enumerable.*

- Suppose both L and $\bar{L}$ are recursively enumerable, accepted by M and $\bar{M}$, respectively.
- Simulate M and $\bar{M}$ in an *interleaved* fashion.
- If M accepts, then halt on state "yes" because $x \in L$.
- If $\bar{M}$ accepts, then halt on state "no" because $x \notin L$.
- Note that either M or $\bar{M}$ (but not both) must accept the input and halt.

A Very Useful Corollary and Its Consequences

Corollary 11 *L is recursively enumerable but not recursive, then $\bar{L}$ is not recursively enumerable.*

- Suppose $\bar{L}$ is recursively enumerable.
- Then both L and $\bar{L}$ are recursively enumerable.
- By Lemma 10 (p. 150), L is recursive, a contradiction.

Corollary 12 *$\bar{H}$ is not recursively enumerable.*[a]

[a] Recall that $\bar{H} = \{ M; x : M(x) = \nearrow \}$.

R, RE, and coRE

RE: The set of all recursively enumerable languages.

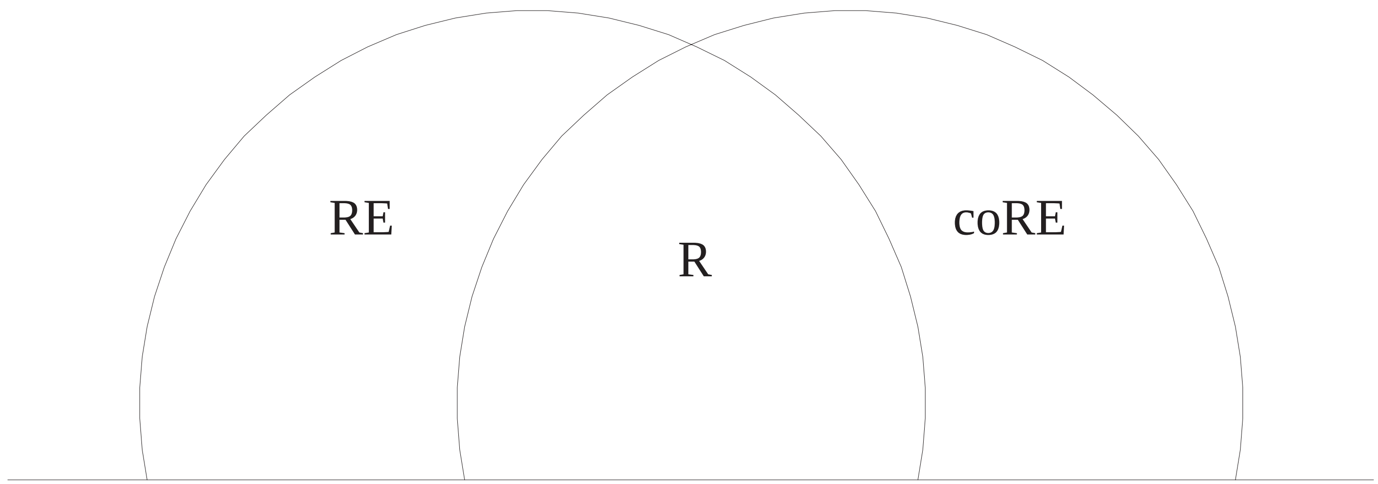
coRE: The set of all languages whose complements are recursively enumerable.

R: The set of all recursive languages.

- Note that coRE is not $\overline{\text{RE}}$.
  - coRE $= \{ L : \overline{L} \in \text{RE} \} = \{ \overline{L} : L \in \text{RE} \}$.
  - $\overline{\text{RE}} = \{ L : L \notin \text{RE} \}$.

R, RE, and coRE (concluded)

- R = RE ∩ coRE (p. 150).
- There exist languages in RE but not in R and not in coRE.
  - Such as H (p. 132, p. 133, and p. 151).
- There are languages in coRE but not in RE.
  - Such as $\bar{H}$ (p. 151).
- There are languages in neither RE nor coRE.


RE
R
coRE

H Is Complete for RE[a]

- Let L be any recursively enumerable language.
- Assume M accepts L.
- Clearly, one can decide whether $x \in L$ by asking if $M ; x \in H$.
- Hence *all* recursively enumerable languages are reducible to H!
- H is said to be **RE-complete**.

[a] Post (1944).

Undecidability in Logic and Mathematics

- First-order logic is undecidable (answer to Hilbert's (1928) *Entscheidungsproblem*).[a]
- Natural numbers with addition and multiplication is undecidable.[b]
- Rational numbers with addition and multiplication is undecidable.[c]

---

[a] Church (1936).
[b] Rosser (1937).
[c] Robinson (1948).

Undecidability in Logic and Mathematics (concluded)

- Natural numbers with addition and equality is decidable and complete.[a]
- Elementary theory of groups is undecidable.[b]

---

[a]Presburger's Master's thesis (1928), his only work in logic. The direction was suggested by Tarski. Mojžesz Presburger (1904–1943) died in a concentration camp during World War II.

[b]Tarski (1949).

Julia Hall Bowman Robinson (1919–1985)

Alfred Tarski (1901–1983)

Boolean Logic

Both of us had said the very same thing.
Did we both speak the truth
—or one of us did
—or neither?
— Joseph Conrad (1857–1924),
Lord Jim (1900)

Boolean Logic[a]

Boolean variables: $x_1, x_2, \ldots$.

Literals: x_i, $\neg x_i$.

Boolean connectives: $\lor, \land, \neg$.

Boolean expressions: Boolean variables, $\neg\phi$ (**negation**), $\phi_1 \lor \phi_2$ (**disjunction**), $\phi_1 \land \phi_2$ (**conjunction**).

- $\bigvee_{i=1}^{n} \phi_i$ stands for $\phi_1 \lor \phi_2 \lor \cdots \lor \phi_n$.
- $\bigwedge_{i=1}^{n} \phi_i$ stands for $\phi_1 \land \phi_2 \land \cdots \land \phi_n$.

Implications: $\phi_1 \Rightarrow \phi_2$ is a shorthand for $\neg\phi_1 \lor \phi_2$.

Biconditionals: $\phi_1 \Leftrightarrow \phi_2$ is a shorthand for $(\phi_1 \Rightarrow \phi_2) \land (\phi_2 \Rightarrow \phi_1)$.

---

[a] George Boole (1815–1864) in 1847.

Truth Assignments

- A **truth assignment** T is a mapping from boolean variables to **truth values** `true` and `false`.
- A truth assignment is **appropriate** to boolean expression ϕ if it defines the truth value for every variable in ϕ.
  - $\{ x_1 = \texttt{true}, x_2 = \texttt{false} \}$ is appropriate to $x_1 \lor x_2$.
  - $\{ x_2 = \texttt{true}, x_3 = \texttt{false} \}$ is not appropriate to $x_1 \lor x_2$.

Satisfaction

- $T \models \phi$ means boolean expression ϕ is true under T; in other words, T **satisfies** ϕ.
- ϕ_1 and ϕ_2 are **equivalent**, written

$$\phi_1 \equiv \phi_2,$$

  if for any truth assignment T appropriate to both of them, $T \models \phi_1$ if and only if $T \models \phi_2$.

Truth Tables

- Suppose ϕ has n boolean variables.
- A **truth table** contains 2^n rows.
- Each row corresponds to one truth assignment of the n variables and records the truth value of ϕ under it.
- A truth table can be used to prove if two boolean expressions are equivalent.
  - Just check if they give identical truth values under all appropriate truth assignments.

A Truth Table

p	q	$p \land q$
0	0	0
0	1	0
1	0	0
1	1	1

A Second Truth Table

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

A Third Truth Table

p	$\neg p$
0	1
1	0

Proof of Equivalency by the Truth Table:

$$p \Rightarrow q \equiv \neg q \Rightarrow \neg p$$

p	q	$p \Rightarrow q$	$\neg q \Rightarrow \neg p$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

De Morgan's Laws[a]

- **De Morgan's laws** say that

$$\begin{aligned} \neg(\phi_1 \land \phi_2) &\equiv \neg\phi_1 \lor \neg\phi_2, \\ \neg(\phi_1 \lor \phi_2) &\equiv \neg\phi_1 \land \neg\phi_2. \end{aligned}$$

- Here is a proof of the first law:

ϕ_1	ϕ_2	$\neg(\phi_1 \land \phi_2)$	$\neg\phi_1 \lor \neg\phi_2$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

[a] Augustus DeMorgan (1806–1871) or William of Ockham (1288–1348).

Conjunctive Normal Forms

- A boolean expression ϕ is in **conjunctive normal form** (**CNF**) if
$$\phi = \bigwedge_{i=1}^{n} C_i,$$
where each **clause** C_i is the disjunction of zero or more literals.[a]
  - For example,
$$(x_1 \lor x_2) \land (x_1 \lor \neg x_2) \land (x_2 \lor x_3).$$
- Convention: An empty CNF is satisfiable, but a CNF containing an empty clause is not.

[a]Improved by Mr. Aufbu Huang (R95922070) on October 5, 2006.

Disjunctive Normal Forms

- A boolean expression ϕ is in **disjunctive normal form** (**DNF**) if

$$\phi = \bigvee_{i=1}^{n} D_i,$$

where each **implicant** D_i is the conjunction of zero or more literals.

  – For example,

$$(x_1 \wedge x_2) \vee (x_1 \wedge \neg x_2) \vee (x_2 \wedge x_3).$$

Clauses and Implicants

- The $\bigvee$ of clauses remains a clause.
  - For example,

$$\begin{aligned} & (x_1 \lor x_2) \lor (x_1 \lor \neg x_2) \lor (x_2 \lor x_3) \\ = \quad & x_1 \lor x_2 \lor x_1 \lor \neg x_2 \lor x_2 \lor x_3. \end{aligned}$$

- The $\bigwedge$ of implicants remains an implicant.
  - For example,

$$\begin{aligned} & (x_1 \land x_2) \land (x_1 \land \neg x_2) \land (x_2 \land x_3) \\ = \quad & x_1 \land x_2 \land x_1 \land \neg x_2 \land x_2 \land x_3. \end{aligned}$$

Any Expression ϕ Can Be Converted into CNFs and DNFs

$\phi = x_j$:

- This is trivially true.

$\phi = \neg\phi_1$ and a CNF is sought:

- Turn ϕ_1 into a DNF.
- Apply de Morgan's laws to make a CNF for ϕ.

$\phi = \neg\phi_1$ and a DNF is sought:

- Turn ϕ_1 into a CNF.
- Apply de Morgan's laws to make a DNF for ϕ.

Any Expression ϕ Can Be Converted into CNFs and DNFs (continued)

$\phi = \phi_1 \lor \phi_2$ **and a DNF is sought:**

- Make ϕ_1 and ϕ_2 DNFs.

$\phi = \phi_1 \lor \phi_2$ **and a CNF is sought:**

- Turn ϕ_1 and ϕ_2 into CNFs,[a]

$$\phi_1 = \bigwedge_{i=1}^{n_1} A_i, \quad \phi_2 = \bigwedge_{j=1}^{n_2} B_j.$$

- Set

$$\phi = \bigwedge_{i=1}^{n_1} \bigwedge_{j=1}^{n_2} (A_i \lor B_j).$$

[a] Corrected by Mr. Chun-Jie Yang (`R99922150`) on November 9, 2010.

Any Expression ϕ Can Be Converted into CNFs and DNFs (concluded)

$\phi = \phi_1 \wedge \phi_2$ and a CNF is sought:

- Make ϕ_1 and ϕ_2 CNFs.

$\phi = \phi_1 \wedge \phi_2$ and a DNF is sought:

- Turn ϕ_1 and ϕ_2 into DNFs,

$$\phi_1 = \bigvee_{i=1}^{n_1} A_i, \quad \phi_2 = \bigvee_{j=1}^{n_2} B_j.$$

- Set

$$\phi = \bigvee_{i=1}^{n_1} \bigvee_{j=1}^{n_2} (A_i \wedge B_j).$$

An Example: Turn $\neg((a \wedge y) \vee (z \vee w))$ into a DNF

$$
\begin{aligned}
& \neg((a \wedge y) \vee (z \vee w)) \\
\stackrel{\neg(\text{CNF}\vee\text{CNF})}{=} \;& \neg(((a) \wedge (y)) \vee ((z \vee w))) \\
\stackrel{\neg(\text{CNF})}{=} \;& \neg((a \vee z \vee w) \wedge (y \vee z \vee w)) \\
\stackrel{\text{de Morgan}}{=} \;& \neg(a \vee z \vee w) \vee \neg(y \vee z \vee w) \\
\stackrel{\text{de Morgan}}{=} \;& (\neg a \wedge \neg z \wedge \neg w) \vee (\neg y \wedge \neg z \wedge \neg w).
\end{aligned}
$$