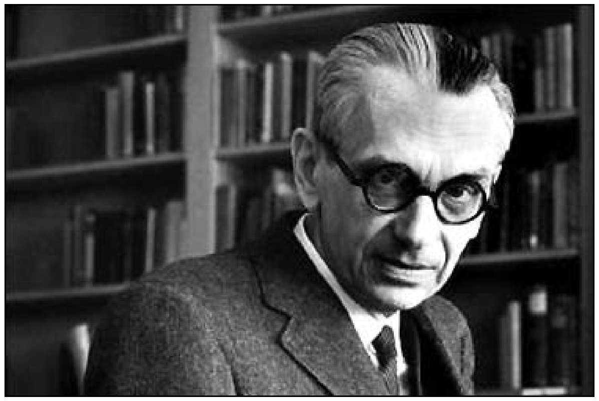
Turing-Computable Functions

- Let $f : (\Sigma - \{\bigsqcup\})^* \to \Sigma^*$.
  - Optimization problems, root finding problems, etc.
- Let M be a TM with alphabet Σ.
- M **computes** f if for any string $x \in (\Sigma - \{\bigsqcup\})^*$, $M(x) = f(x)$.
- We call f a **recursive function**[a] if such an M exists.

[a]Kurt Gödel (1931, 1934).

Kurt Gödel[a] (1906–1978)

Quine (1978), "this theorem [$\cdots$] sealed his immortality."



[a]This photo was taken by Alfred Eisenstaedt (1898–1995).

Church's Thesis or the Church-Turing Thesis

- What is computable is Turing-computable; TMs are algorithms.[a]
- No "intuitively computable" problems have been shown not to be Turing-computable, yet.[b]

[a]Church (1935); Kleene (1953).

[b]Quantum computer of Manin (1980) and Feynman (1982) and DNA computer of Adleman (1994).

Church's Thesis or the Church-Turing Thesis (concluded)

- Many other computation models have been proposed.
  - Recursive function (Gödel), λ calculus (Church), formal language (Post), assembly language-like RAM (Shepherdson & Sturgis), boolean circuits (Shannon), extensions of the Turing machine (more strings, two-dimensional strings, and so on), etc.
- All have been proved to be equivalent.

Alonso Church (1903–1995)

Extended Church's Thesis[a]

- All "reasonably succinct encodings" of problems are *polynomially related* (e.g., n^2 vs. n^6).
  - Representations of a graph as an adjacency matrix and as a linked list are both succinct.
  - The *unary* representation of numbers is not succinct.
  - The *binary* representation of numbers is succinct.
    - 1001_2 vs. 111111111_1.
- All numbers for TMs will be binary from now on.

[a]Some call it "polynomial Church's thesis," which Lószló Lovász attributed to Leonid Levin.

Extended Church's Thesis (concluded)

- Representations that are not succinct may give misleadingly low complexities.
  - Consider an algorithm with binary inputs that runs in 2^n steps.
  - Suppose the input uses unary representation instead.
  - Then the same algorithm runs in linear time because the input length is now 2^n!
- So a succinct representation means honest accounting.

Physical Church-Turing Thesis

- Church's thesis

  is a profound claim about the physical laws of our universe, i.e.: any physical system that purports to be a 'computer' is not capable of any computational task that a Turing machine is incapable of.[a]

- Church's and extended Church's theses

  are not statements about mathematics, but rather conjectured constraints on physical laws.[b]

---

[a] Warren Smith (1998).
[b] Yao (2003).

Physical Church-Turing Thesis (concluded)

- The **physical Church-Turing thesis** states that:

  Anything computable in physics can also be computed on a Turing machine.[a]

- The universe is a Turing machine.[b]

---

[a]Cooper (2012).

[b]Edward Fredkin's (1992) controversial digital physics.

The Strong Church-Turing Thesis[a]

- The **strong Church-Turing thesis** states that:

  A Turing machine can compute *any* function computable by any "reasonable" physical device with only polynomial slowdown.[b]

- A CPU and a DSP chip are good examples of physical devices.[c]

[a] Vergis, Steiglitz, and Dickinson (1986).

[b] `http://ocw.mit.edu/courses/mathematics/18-405j-advanced-complexity-theory-fall-2001/lecture-notes/lecture10.pdf`

[c] Thanks to a lively discussion on September 23, 2014.

The Strong Church-Turing Thesis (concluded)

- Factoring is believed to be a hard problem for Turing machines (but there is no proof yet).
- But a quantum computer can factor numbers in probabilistic polynomial time.[a]
- So if a large-scale quantum computer can be reliably built, the strong Church-Turing thesis may be refuted.[b]

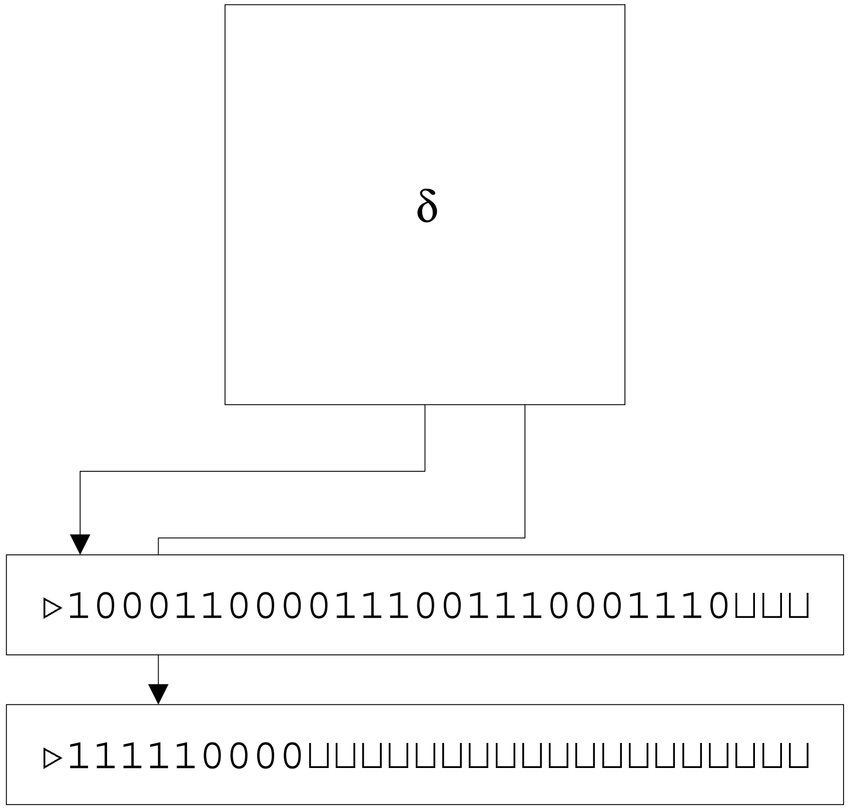
[a]Shor (1994).

[b]Contributed by Mr. Kai-Yuan Hou (B99201038, R03922014) on September 22, 2015.

Turing Machines with Multiple Strings

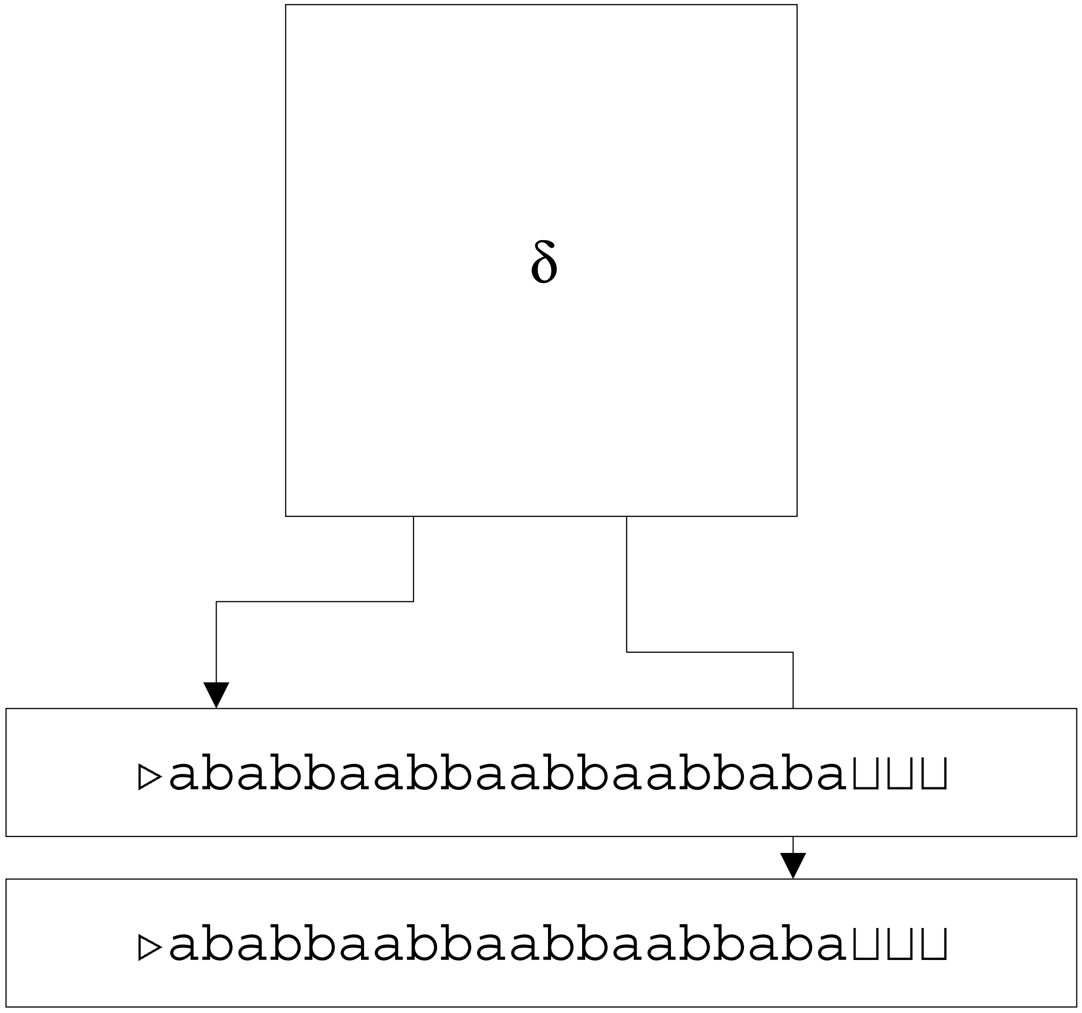
- A k-string Turing machine (TM) is a quadruple $M = (K, \Sigma, \delta, s)$.
- K, Σ, s are as before.
- $\delta : K \times \Sigma^k \to (K \cup \{h, \text{“yes”}, \text{“no”}\}) \times (\Sigma \times \{\leftarrow, \rightarrow, -\})^k$.
- All strings start with a $\rhd$.
- The first string contains the input.
- Decidability and acceptability are the same as before.
- When TMs compute functions, the output is the last (kth) string.

A 2-String TM



PALINDROME Revisited

- A 2-string TM can decide PALINDROME in $O(n)$ steps.
  - It copies the input to the second string.
  - The cursor of the first string is positioned at the first symbol of the input.
  - The cursor of the second string is positioned at the last symbol of the input.
  - The symbols under the cursors are then compared.
  - The two cursors are then moved in opposite directions until the ends are reached.
  - The machine accepts if and only if the symbols under the two cursors are identical at all steps.


δ
▷ababbaabbaabbaabbaba⊔⊔⊔
▷ababbaabbaabbaabbaba⊔⊔⊔

PALINDROME Revisited (concluded)

- The running times of a 2-string TM and a single-string TM are quadratically related: n^2 vs. n.
- This is consistent with the extended Church's thesis.
  - "Reasonable" models are related polynomially in running times.

Configurations and Yielding

- The concept of configuration and yielding is the same as before except that a configuration is a $(2k+1)$-tuple

$$(q, w_1, u_1, w_2, u_2, \ldots, w_k, u_k).$$

  - $w_i u_i$ is the ith string.
  - The ith cursor is reading the last symbol of w_i.
  - Recall that $\rhd$ is each w_i's first symbol.

- The k-string TM's initial configuration is

$$(s, \overbrace{\underbrace{\rhd, x}_{1}, \underbrace{\rhd, \epsilon}_{2}, \underbrace{\rhd, \epsilon}_{3}, \ldots, \underbrace{\rhd, \epsilon}_{k}}^{2k}).$$

Time seemed to be
the most obvious measure
of complexity.
— Stephen Arthur Cook (1939–)

Time Complexity

- The multistring TM is the basis of our notion of the time expended by TMs.
- If a k-string TM M halts after t steps on input x, then the **time required by M on input x** is t.
- If $M(x) = \nearrow$, then the time required by M on x is ∞.

Time Complexity (concluded)

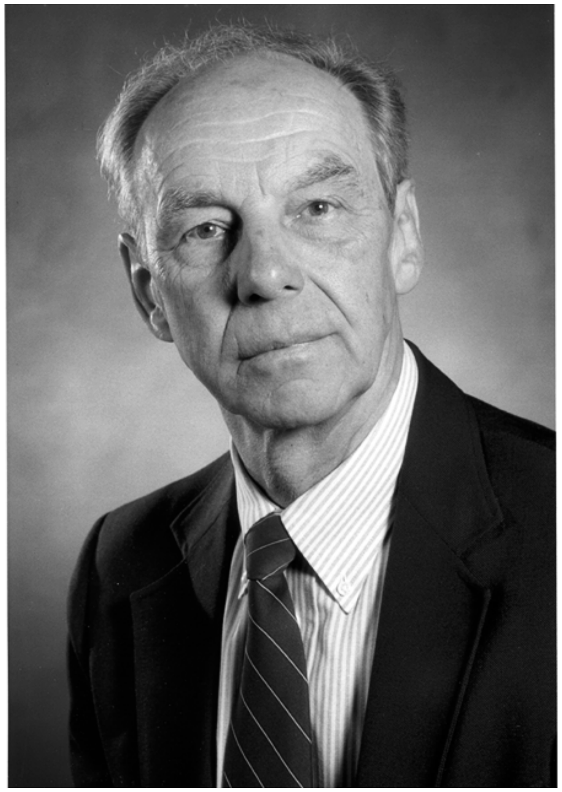
- Machine M **operates within time** $f(n)$ for $f : \mathbb{N} \to \mathbb{N}$ if for any input string x, the time required by M on x is at most $f(|\,x\,|)$.
  - $|\,x\,|$ is the length of string x.
- Function $f(n)$ is a **time bound** for M.

Time Complexity Classes[a]

- Suppose language $L \subseteq (\Sigma - \{\sqcup\})^*$ is decided by a multistring TM operating in time $f(n)$.
- We say $L \in \text{TIME}(f(n))$.
- $\text{TIME}(f(n))$ is the set of languages decided by TMs with multiple strings operating within time bound $f(n)$.
- $\text{TIME}(f(n))$ is a **complexity class**.
  - PALINDROME is in $\text{TIME}(f(n))$, where $f(n) = O(n)$.
- Trivially, $\text{TIME}(f(n)) \subseteq \text{TIME}(g(n))$ if $f(n) \leq g(n)$ for all n.

[a]Hartmanis and Stearns (1965); Hartmanis, Lewis, and Stearns (1965).

Juris Hartmanis[a] (1928–)



[a]Turing Award (1993).

Richard Edwin Stearns[a] (1936–)

[a]Turing Award (1993).

The Simulation Technique

Theorem 3 *Given any k-string M operating within time $f(n)$, there exists a (single-string) M' operating within time $O(f(n)^2)$ such that $M(x) = M'(x)$ for any input x.*

- The single string of M' implements the k strings of M.

The Proof

- Represent configuration $(q, w_1, u_1, w_2, u_2, \ldots, w_k, u_k)$ of M by this string of M':

$$(q, \rhd w_1' u_1 \lhd w_2' u_2 \lhd \cdots \lhd w_k' u_k \lhd \lhd).$$

  - $\lhd$ is a special delimiter.
  - w_i' is w_i with the first[a] and last symbols "primed."
  - It serves the purpose of "," in a configuration.[b]

[a]The first symbol is always $\rhd$.

[b]An alternative is to use $(q, \rhd w_1' u_1,' \lhd w_2',' u_2 \lhd \cdots \lhd w_k',' u_k \lhd \lhd)$ by priming only $\rhd$ in w_i, where ",′" is a new symbol.

The Proof (continued)

- The "priming" of the last symbol of w_i ensures that M' knows which symbol is under each cursor of M.[a]
- The first symbol of w_i is the primed version of $\triangleright$: $\triangleright'$.
  - Recall TM cursors are not allowed to move to the left of $\triangleright$ (p. 21).
  - Now the cursor of M' can move *between* the simulated strings of M.[b]

[a]Added because of comments made by Mr. Che-Wei Chang (R95922093) on September 27, 2006.

[b]Thanks to a lively discussion on September 22, 2009.

The Proof (continued)

- The initial configuration of M' is

$$(s, \rhd \rhd'' \, x \lhd \overbrace{\rhd'' \lhd \cdots \rhd'' \lhd}^{k-1 \text{ pairs}} \lhd).$$

  - $\rhd''$ is double-primed because it is the beginning and the ending symbol as the cursor is reading it.[a]

[a] Added after the class discussion on September 20, 2011.

The Proof (continued)

- We simulate each move of M thus:
  1. M' scans the string to pick up the k symbols under the cursors.
     - The states of M' must be enlarged to include $K \times \Sigma^k$ to remember them.[a]
     - The transition functions of M' must also reflect it.
  2. M' then changes the string to reflect the overwriting of symbols and cursor movements of M.

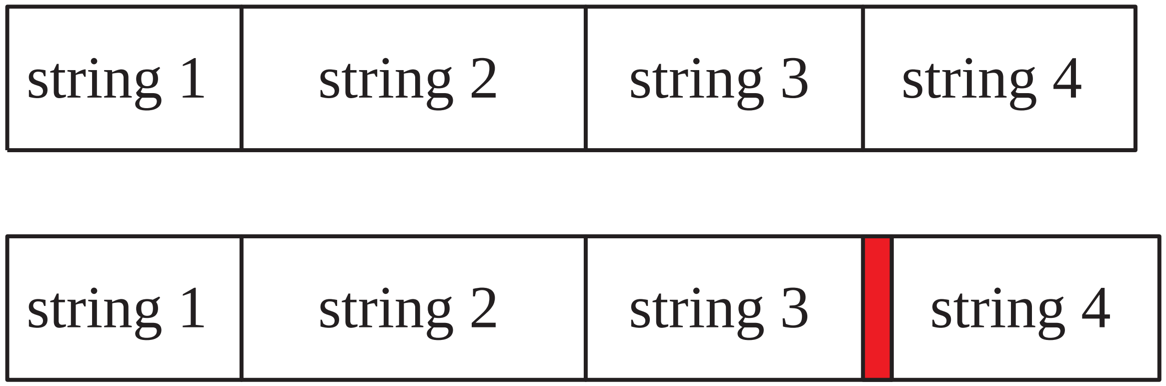
[a]Recall the TM program on p. 28.

The Proof (continued)

- It is possible that some strings of M need to be lengthened (see next page).
  - The linear-time algorithm on p. 34 can be used for each such string.
- The simulation continues until M halts.
- M' then erases all strings of M except the last one.[a]

[a]Because whatever appears on the string of M' will be considered the output. So those $\triangleright'$s and $\triangleright''$s need to be removed.

The Proof (continued)[a]



[a]If we interleave the strings, the simulation may be easier. Contributed by Mr. Kai-Yuan Hou (`B99201038`, `R03922014`) on September 22, 2015. This is similar to constructing a single-string *multi-track* TM in, e.g., Hopcroft and Ullman (1969).

The Proof (continued)

- Since M halts within time $f(|x|)$, none of its strings ever becomes longer than $f(|x|)$.[a]
- The length of the string of M' at any time is $O(kf(|x|))$.
- Simulating each step of M takes, *per string of* M, $O(kf(|x|))$ steps.
  - $O(f(|x|))$ steps to collect information from this string.
  - $O(kf(|x|))$ steps to write and, if needed, to lengthen the string.

[a]We tacitly assume $f(n) \geq n$.

The Proof (concluded)

- M' takes $O(k^2 f(|x|))$ steps to simulate each step of M because there are k strings.
- As there are $f(|x|)$ steps of M to simulate, M' operates within time $O(k^2 f(|x|)^2)$.[a]

[a]Is the time reduced to $O(kf(|x|)^2)$ if the interleaving data structure is adopted?

Simulation with Two-String TMs

We can do better with two-string TMs.

Theorem 4 *Given any k-string M operating within time $f(n)$, $k > 2$, there exists a two-string M' operating within time $O(f(n) \log f(n))$ such that $M(x) = M'(x)$ for any input x.*

Linear Speedup[a]

Theorem 5 *Let $L \in \mathit{TIME}(f(n))$. Then for any $\epsilon > 0$, $L \in \mathit{TIME}(f'(n))$, where $f'(n) = \epsilon f(n) + n + 2$.*

[a]Hartmanis and Stearns (1965).

Implications of the Speedup Theorem

- State size can be traded for speed.[a]
- If the running time is cn with $c > 1$, then c can be made arbitrarily close to 1.
- If the running time is superlinear, say $14n^2 + 31n$, then the constant in the leading term (14 in this example) can be made arbitrarily small.
  - *Arbitrary* linear speedup can be achieved.[b]
  - This justifies the big-O notation in the analysis of algorithms.

[a] $m^k \cdot |\Sigma|^{3mk}$-fold increase to gain a speedup of $O(m)$. No free lunch.

[b] Can you apply the theorem multiple times to achieve superlinear speedup? Thanks to a question by a student on September 21, 2010.

P

- By the linear speedup theorem, any polynomial time bound can be represented by its leading term n^k for some $k \geq 1$.
- If $L \in \text{TIME}(n^k)$ for some $k \in \mathbb{N}$, it is a **polynomially decidable language**.
  - Clearly, $\text{TIME}(n^k) \subseteq \text{TIME}(n^{k+1})$.
- The union of all polynomially decidable languages is denoted by P:
$$\text{P} = \bigcup_{k>0} \text{TIME}(n^k).$$
- P contains problems that can be efficiently solved.

Philosophers have explained space.
They have not explained time.
— Arnold Bennett (1867–1931),
How To Live on 24 Hours a Day (1910)

I keep bumping into that silly quotation
attributed to me that says
640K of memory is enough.
— Bill Gates (1996)

Space Complexity

- Consider a k-string TM M with input x.
- Assume non-$\sqcup$ is never written over by $\sqcup$.[a]
  - The purpose is not to artificially reduce the space needs (see below).
- If M halts in configuration

$$(H, w_1, u_1, w_2, u_2, \ldots, w_k, u_k),$$

then the **space required by M on input x** is

$$\sum_{i=1}^{k} |\, w_i u_i \,|.$$

[a]Corrected by Ms. Chuan-Ju Wang (`R95922018`, `F95922018`) on September 27, 2006.

Space Complexity (continued)

- Suppose we do not charge the space used only for input and output.
- Let $k > 2$ be an integer.
- A **k-string Turing machine with input and output** is a k-string TM that satisfies the following conditions.
  - The input string is *read-only*.[a]
  - The last string, the output string, is *write-only*.
    - So the cursor never moves to the left.
  - The cursor of the input string does not wander off into the $\sqcup$s.

[a]Called an **off-line** TM in Hartmanis, Lewis, and Stearns (1965).

Space Complexity (concluded)

- If M is a TM with input and output, then the space required by M on input x is
$$\sum_{i=2}^{k-1} |\, w_i u_i \,|.$$
- Machine M **operates within space bound** $f(n)$ for $f : \mathbb{N} \to \mathbb{N}$ if for any input x, the space required by M on x is at most $f(|\, x \,|)$.

Space Complexity Classes

- Let L be a language.
- Then
$$L \in \text{SPACE}(f(n))$$
  if there is a TM with input and output that decides L and operates within space bound $f(n)$.
- SPACE$(f(n))$ is a set of languages.
  - PALINDROME $\in$ SPACE$(\log n)$.[a]
- As in the linear speedup theorem (p. 90), constant coefficients do not matter.

[a]Keep 3 counters.

Nondeterminism[a]

- A **nondeterministic Turing machine** (**NTM**) is a quadruple $N = (K, \Sigma, \Delta, s)$.
- K, Σ, s are as before.
- $\Delta \subseteq K \times \Sigma \times (K \cup \{h, \text{“yes”}, \text{“no”}\}) \times \Sigma \times \{\leftarrow, \rightarrow, -\}$ is a relation, not a function.[b]
  - For each state-symbol combination (q, σ), there may be multiple valid next steps.
  - Multiple lines of code may be applicable.

[a] Rabin and Scott (1959).

[b] Corrected by Mr. Jung-Ying Chen (D95723006) on September 23, 2008.

Nondeterminism (continued)

- As before, a program contains lines of code:

$$\begin{array}{rcl} (q_1, \sigma_1, p_1, \rho_1, D_1) & \in & \Delta, \\ (q_2, \sigma_2, p_2, \rho_2, D_2) & \in & \Delta, \\ & \vdots & \\ (q_n, \sigma_n, p_n, \rho_n, D_n) & \in & \Delta. \end{array}$$

  – We cannot write

$$\delta(q_i, \sigma_i) = (p_i, \rho_i, D_i)$$

    as in the deterministic case (p. 22) anymore.

Nondeterminism (concluded)

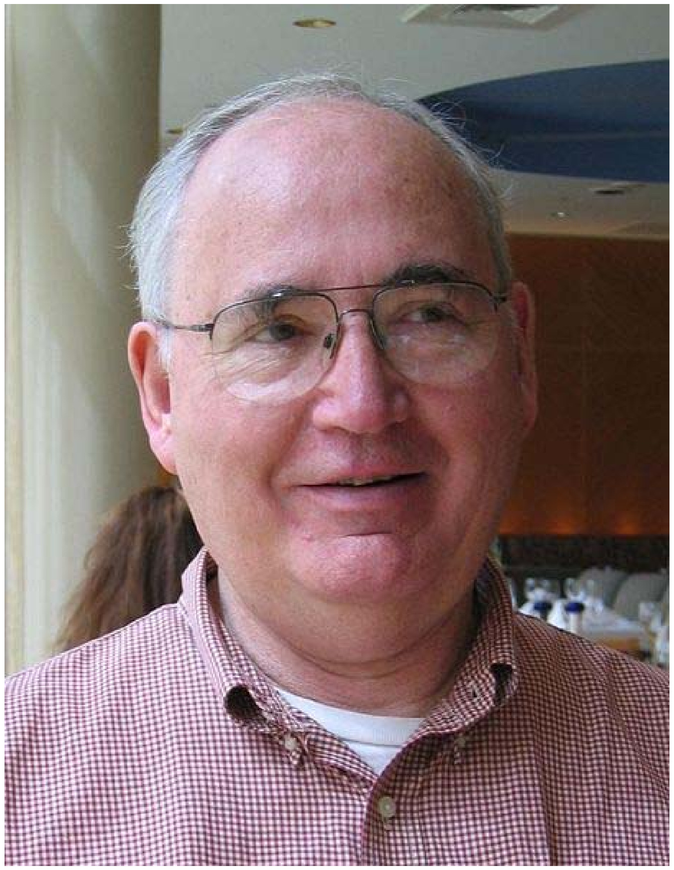
- A configuration yields another configuration in one step if there *exists* a rule in Δ that makes this happen.
- But only one will be taken.
- So there is only a single thread of computation.[a]
  - Nondeterminism is not parallelism, multiprocessing, or multithreading.

[a]Thanks to a lively discussion on September 22, 2015.

Michael O. Rabin[a] (1931–)

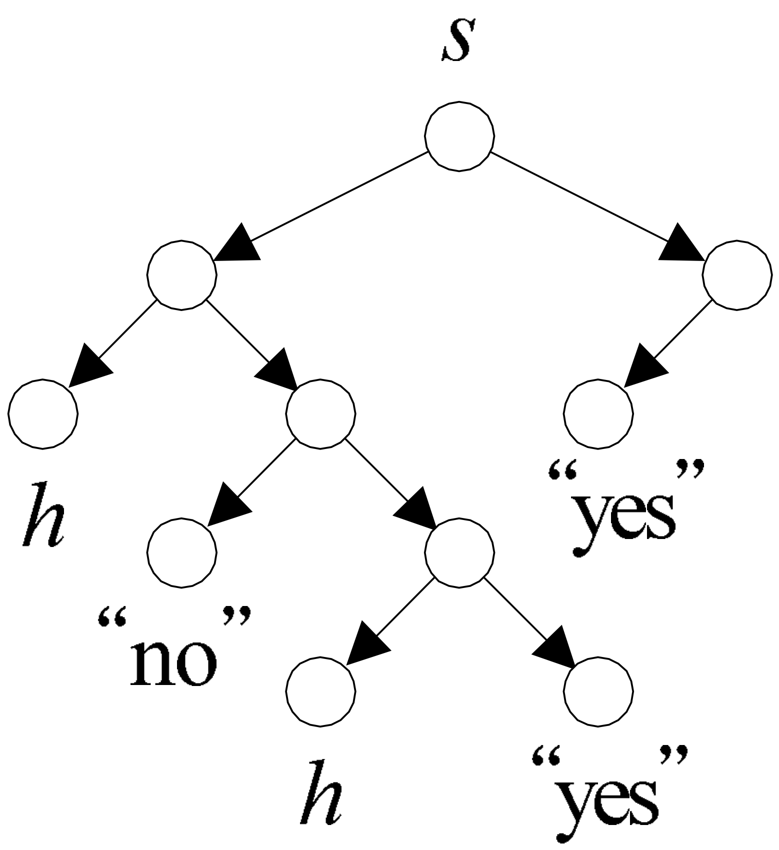
[a]Turing Award (1976).

Dana Stewart Scott[a] (1932–)



[a]Turing Award (1976).

Computation Tree and Computation Path



Decidability under Nondeterminism

- Let L be a language and N be an NTM.
- N **decides** L if for any $x \in \Sigma^*$, $x \in L$ if and only if there is a sequence of valid configurations that ends in "yes."
- In other words,
  - If $x \in L$, then $N(x) =$ "yes" for some computation path.
  - If $x \notin L$, then $N(x) \neq$ "yes" for all computation paths.

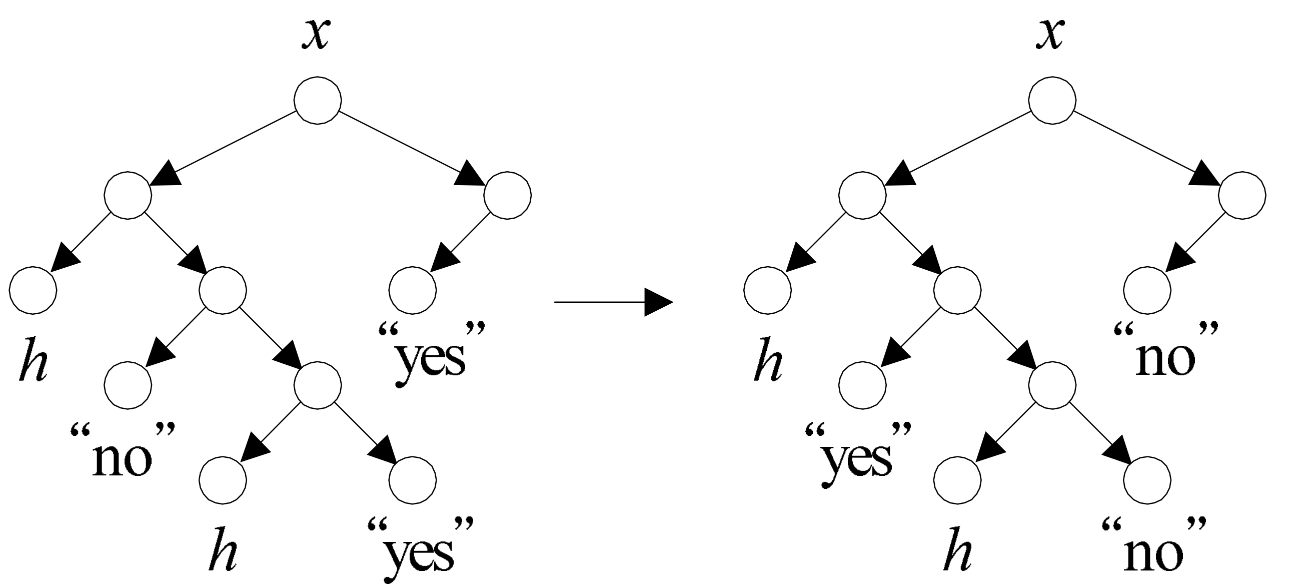
Decidability under Nondeterminism (concluded)

- It is not required that the NTM halts in all computation paths.[a]
- If $x \notin L$, no nondeterministic choices should lead to a "yes" state.
- The key is the algorithm's *overall* behavior not whether it gives a correct answer for each particular run.
- Note that determinism is a special case of nondeterminism.

[a]So "accepts" is a more proper term, and other books use "decides" only when the NTM always halts.

Complementing a TM's Halting States

- Let M decide L, and M' be M after "yes" $\leftrightarrow$ "no".
- If M is a deterministic TM, then M' decides $\bar{L}$.
  - So M and M' decide languages that complement each other.
- But if M is an NTM, then M' may not decide $\bar{L}$.
  - It is possible that M and M' accept the same input x (see next page).
  - So M and M' accept languages that are not complements of each other.



x
h
"no"
h
"yes"
"yes"
x
h
"yes"
h
"no"
"no"

Time Complexity under Nondeterminism

- Nondeterministic machine N decides L **in time** $f(n)$, where $f : \mathbb{N} \to \mathbb{N}$, if
  - N decides L, and
  - for any $x \in \Sigma^*$, N does not have a computation path longer than $f(|\, x\,|)$.
- We charge only the "depth" of the computation tree.

Time Complexity Classes under Nondeterminism

- NTIME($f(n)$) is the set of languages decided by NTMs within time $f(n)$.
- NTIME($f(n)$) is a complexity class.

NP ("Nondeterministic Polynomial")

- Define
$$\mathrm{NP} = \bigcup_{k>0} \mathrm{NTIME}(n^k).$$
- Clearly $\mathrm{P} \subseteq \mathrm{NP}$.
- Think of NP as efficiently *verifiable* problems (see p. 334).
  - Boolean satisfiability (p. 114 and p. 200).
- The most important open problem in computer science is whether $\mathrm{P} = \mathrm{NP}$.

Simulating Nondeterministic TMs

Nondeterminism does not add power to TMs.

Theorem 6 *Suppose language L is decided by an NTM N in time $f(n)$. Then it is decided by a 3-string deterministic TM M in time $O(c^{f(n)})$, where $c > 1$ is some constant depending on N.*

- On input x, M goes down every computation path of N using depth-first search.
  - M does *not* need to know $f(n)$.
  - As N is time-bounded, the depth-first search will not run indefinitely.

The Proof (concluded)

- If any path leads to "yes," then M immediately enters the "yes" state.
- If none of the paths leads to "yes," then M enters the "no" state.
- The simulation takes time $O(c^{f(n)})$ for some $c > 1$ because the computation tree has that many nodes.

Corollary 7 $\text{NTIME}(f(n))) \subseteq \bigcup_{c>1} \text{TIME}(c^{f(n)})$.[a]

[a]Mr. Kai-Yuan Hou (`B99201038`, `R03922014`) on October 6, 2015: $\bigcup_{c>1} \text{TIME}(c^{f(n)}) \subseteq \text{NTIME}(f(n)))$?

NTIME vs. TIME

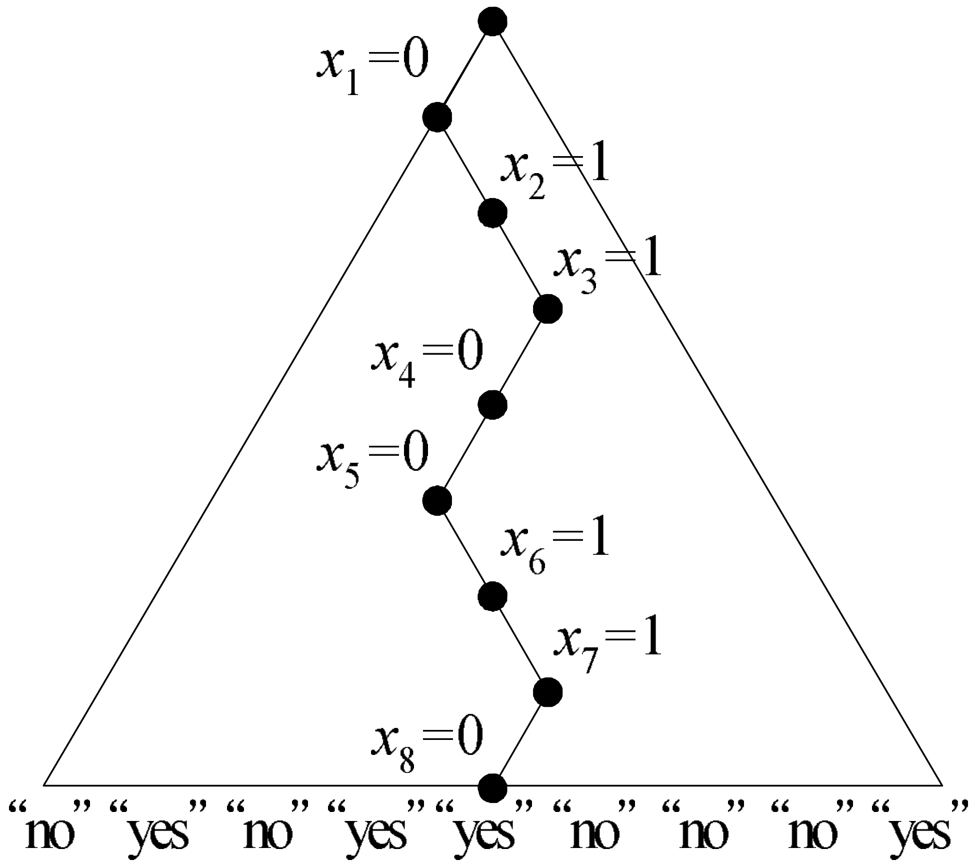
- Does converting an NTM into a TM require exploring all computation paths of the NTM as done in Theorem 6 (p. 111)?
- This is a key question in theory with important practical implications.

A Nondeterministic Algorithm for Satisfiability

ϕ is a boolean formula with n variables.

```
1: **for** $i = 1, 2, \ldots, n$ **do**
2:     Guess $x_i \in \{ 0, 1 \}$; {Nondeterministic choices.}
3: **end for**
4: {Verification:}
5: **if** $\phi(x_1, x_2, \ldots, x_n) = 1$ **then**
6:     "yes";
7: **else**
8:     "no";
9: **end if**
```

Computation Tree for Satisfiability



Analysis

- The computation tree is a complete binary tree of depth n.
- Every computation path corresponds to a particular truth assignment[a] out of 2^n.
- ϕ is satisfiable if and only if there is a truth assignment that satisfies ϕ.

[a]Or a sequence of nondeterministic choices.

Analysis (concluded)

- The algorithm decides language

$$\{\,\phi : \phi \text{ is satisfiable}\,\}.$$

  - Suppose ϕ is satisfiable.
    - There is a truth assignment that satisfies ϕ.
    - So there is a computation path that results in "yes."
  - Suppose ϕ is not satisfiable.
    - That means every truth assignment makes ϕ false.
    - So every computation path results in "no."
- General paradigm: Guess a "proof" then verify it.

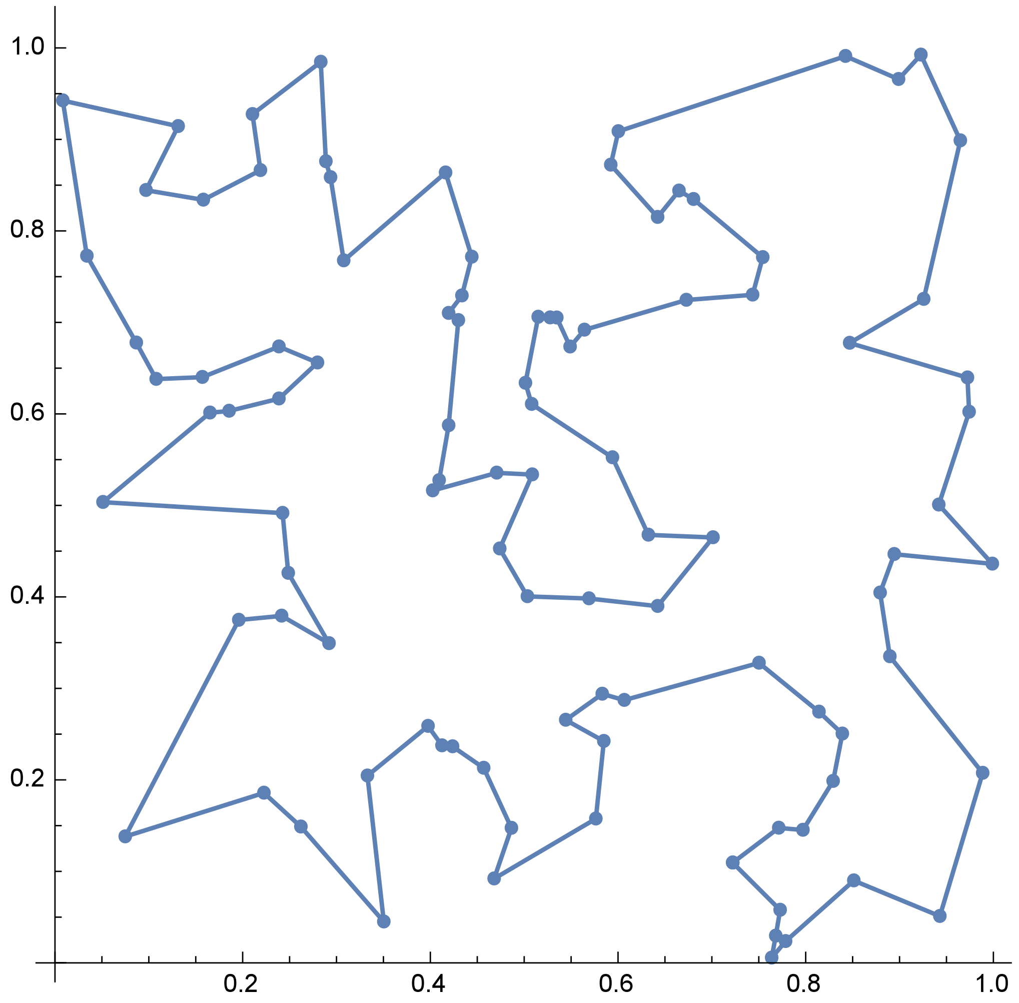
The Traveling Salesman Problem

- We are given n cities $1, 2, \ldots, n$ and integer distance d_{ij} between any two cities i and j.
- Assume $d_{ij} = d_{ji}$ for convenience.
- The **traveling salesman problem** (TSP) asks for the total distance of the shortest tour of the cities.[a]
- The decision version TSP (D) asks if there is a tour with a total distance at most B, where B is an input.[b]

[a]Each city is visited exactly once.

[b]Both problems are extremely important and are equally hard (p. 398 and p. 502).

A Shortest Path



A Nondeterministic Algorithm for TSP (D)

```
1:  **for** $i = 1, 2, \ldots, n$ **do**
2:      Guess $x_i \in \{ 1, 2, \ldots, n \}$; {The $i$th city.}[a]
3:  **end for**
4:  $x_{n+1} := x_1$;
5:  {Verification:}
6:  **if** $x_1, x_2, \ldots, x_n$ are distinct and $\sum_{i=1}^{n} d_{x_i, x_{i+1}} \leq B$ **then**
7:      "yes";
8:  **else**
9:      "no";
10: **end if**
```

[a]Can be made into a series of $\log_2 n$ *binary* choices for each x_i so that the next-state count (2) is a constant, independent of input size. Contributed by Mr. Chih-Duo Hong (`R95922079`) on September 27, 2006.

Analysis

- Suppose the input graph contains at least one tour of the cities with a total distance at most B.
  - Then there is a computation path for that tour.[a]
  - And it leads to "yes."
- Suppose the input graph contains no tour of the cities with a total distance at most B.
  - Then every computation path leads to "no."

[a]It does not mean the algorithm will follow that path. It just means such a computation path (i.e., a sequence of nondeterministic choices) exists.

Remarks on the P $\overset{?}{=}$ NP Open Problem[a]

- Many practical applications depend on answers to the P $\overset{?}{=}$ NP question.
- Verification of password should be easy (so it is in NP).
  - A computer should not take a long time to let a user log in.
- A password system should be hard to crack (loosely speaking, cracking it should not be in P).
- It took logicians 63 years to settle the Continuum Hypothesis; how long will it take for this one?

[a]Contributed by Mr. Kuan-Lin Huang (`B96902079`, `R00922018`) on September 27, 2011.