Theory of Computation

homework 3
Due: 11/3/2015

Problem 1 Denote $L(M)$ as the language L accepted by the Turing machine (TM) M. Determine if the following languages are decidable and explain.

(a) $L_1 = \{M | M \text{ is a TM and there exists an input on which } M \text{ halts within } |M| \text{ steps}\}$.

(b) $L_2 = \{M | M \text{ is a TM and } L(M) \text{ is uncountable}\}$.

Ans:

(a) L_1 is decidable. First, note that there is no need to consider inputs of length greater than $|M|$. If M halts within $|M|$ steps with these inputs, M does not read the whole input before it halts. Then the part of the input that M reads (which is the prefix of the original input and has a length at most $|M|$) will be considered in the algorithm below. So it is sufficient to consider inputs of length at most $|M|$. Next, we construct a TM M' such that it runs M on all inputs of length at most $|M|$. M' halts at "yes" if M accepts at least one of the strings within $|M|$ steps, otherwise M' halts at "no".

(b) L_2 is decidable. In fact, there is no uncountable language over finite alphabets and finite-length strings. L_2 is an empty set.

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Problem 2 Given

$$L_3 = \{M \,;\, x \,;\, y \mid M(x) = y\}.$$

where M is a Turing machine (TM), x and y are strings. Use reduction to show that L_3 is undecidable. (You are not allowed to cite Rice's theorem.)

Ans: We reduce the halting problem H to L_3. Given an instance $M; x$, we construct a TM M_3 such that

$$M_3\,(x) = \begin{cases} 0\,, & \text{if } M(x) \neq \nearrow, \\ \nearrow\,, & \text{otherwise.} \end{cases}$$

Then $M_3\,; x\,; 0 \in L_3$ if and only if $M; x \in H$. Hence, L_3 is undecidable. ■