Theory of Computation Lecture Notes

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Class Information

- Papadimitriou. *Computational Complexity*. 2nd printing. Addison-Wesley. 1995.
  - We more or less follow the topics of the book.
  - Extra materials may be added.
- You may want to review discrete mathematics.

Class Information (concluded)

- More information and lecture notes can be found at

  `www.csie.ntu.edu.tw/~lyuu/complexity.html`

  - Homeworks, exams, solutions and teaching assistants will be announced there.
- Please ask many questions in class.
  - This is the best way for me to remember you in a large class.[a]

[a] "[A] science concentrator [...] said that in his eighth semester of [Harvard] college, there was not a single science professor who could identify him by name." (*New York Times*, September 3, 2003.)

Grading

- Homeworks.
  - Do not copy others' homeworks.
  - Do not give your homeworks for others to copy.
- Two to three exams.
- You must show up for the exams in person.
- If you cannot make it to an exam for a legitimate reason, please email me or a TA beforehand to the extent possible.
- Missing the final exam will automatically earn a "fail" grade.

Problems and Algorithms

I have never done anything "useful."
— Godfrey Harold Hardy (1877–1947),
A Mathematician's Apology (1940)

What This Course Is All About

Computation: What is computation?

Computability: What can be computed?

- There are *well-defined* problems that cannot be computed.
- In fact, most problems cannot be computed.

What This Course Is All About (continued)

Complexity: What is a computable problem's inherent complexity?

- Some computable problems require at least exponential time and/or space.
  - They are said to be **intractable**.
- Some practical problems require superpolynomial[a] resources unless certain conjectures are disproved.
- Resources besides time and space: Circuit size, circuit layout area, program size, number of random bits, etc.

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[a]The prefix "super" means "above, beyond."

What This Course Is All About (concluded)

Applications: Intractability results can be very useful.

- Cryptography and security.
- Approximations.
- Conjectures about nature.

Tractability and Intractability

- Tractability means polynomial in terms of the input size n.
  - n, $n \log n$, n^2, n^{90}.
- It results in a fruitful and practical theory of complexity.
- Few practical, tractable problems require a large degree.
- Superpolynomial-time algorithms are seldom practical.
  - $n^{\log n}$, $2^{\sqrt{n}}$,[a] 2^n, $n! \sim \sqrt{2\pi n}\,(n/e)^n$.

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[a] Size of depth-3 circuits to compute the majority function (Wolfovitz (2006)) and certain stochastic models used in finance (Dai (`R86526008`, `D8852600`) and Lyuu (2007); Lyuu and Wang (`F95922018`) (2011); Chiu (`R98723059`) (2012)).

Exponential Growth of *E. Coli*[a]

- Under ideal conditions, *E. Coli* bacteria divide every 20 minutes.
- In two days, a single *E. Coli* bacterium would become 2^{144} bacteria.
- They would weigh 2,664 times the Earth!

[a] Nick Lane, *Power, Sex, Suicide: Mitochondria and the Meaning of Life* (2005).

Growth of Factorials

n	$n!$	n	$n!$
1	1	9	362,880
2	2	10	3,628,800
3	6	11	39,916,800
4	24	12	479,001,600
5	120	13	6,227,020,800
6	720	14	87,178,291,200
7	5040	15	1,307,674,368,000
8	40320	16	20,922,789,888,000

Moore's Law[a] to the Rescue?[b]

- One version of Moore's law says the computing power doubles every 1.5 years.
- So the computing power grows like

$$4^{y/3},$$

  where y is the number of years from now.
- Assume Moore's law holds forever.
- Can you let the law take care of exponential complexity?

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[a] Moore (1965).

[b] Contributed by Ms. Amy Liu (`J94922016`) on May 15, 2006. Thanks also to a lively discussion on September 14, 2010.

Moore's Law to the Rescue (continued)?

- Suppose a problem takes a^n seconds of CPU time to solve now, where n is the input length.
- The same problem will take

$$\frac{a^n}{4^{y/3}}$$

  seconds to solve y years from now.
- In particular, the hardware $3n \log_4 a$ years from now takes 1 second to solve it.
- The overall complexity becomes linear in n!

Moore's Law to the Rescue (concluded)?

- Potential objections:
  - Moore's law may not hold forever.
  - The total number of operations is the same; so the *algorithm* remains exponential in complexity.[a]
- What is a "good" theory on computational complexity?

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[a]Contributed by Mr. Hung-Jr Shiu (`D00921020`) on September 14, 2011.

Turing Machines

Tarski has stressed in his lecture
(and I think justly)
the great importance of
the concept of general recursiveness
(or Turing's computability).
— Kurt Gödel (1946)

What Is Computation?

- That can be coded in an **algorithm**.[a]
- An algorithm is a detailed step-by-step method for solving a problem.
  - The Euclidean algorithm for the greatest common divisor is an algorithm.
  - "Let s be the least upper bound of compact set A" is not an algorithm.
  - "Let s be a smallest element of a finite-sized array" can be solved by an algorithm.

[a]Muhammad ibn Mūsā Al-Khwārizmī (780–850).

Turing Machines[a]

- A Turing machine (TM) is a quadruple $M = (K, \Sigma, \delta, s)$.
- K is a finite set of **states**.[b]
- $s \in K$ is the **initial state**.
- Σ is a finite set of **symbols** (disjoint from K).
  - Σ includes $\sqcup$ (blank) and $\triangleright$ (first symbol).
- $\delta : K \times \Sigma \to (K \cup \{h, \text{“yes”}, \text{“no”}\}) \times \Sigma \times \{\leftarrow, \to, -\}$ is a **transition function**.
  - $\leftarrow$ (left), $\to$ (right), and $-$ (stay) signify cursor movements.

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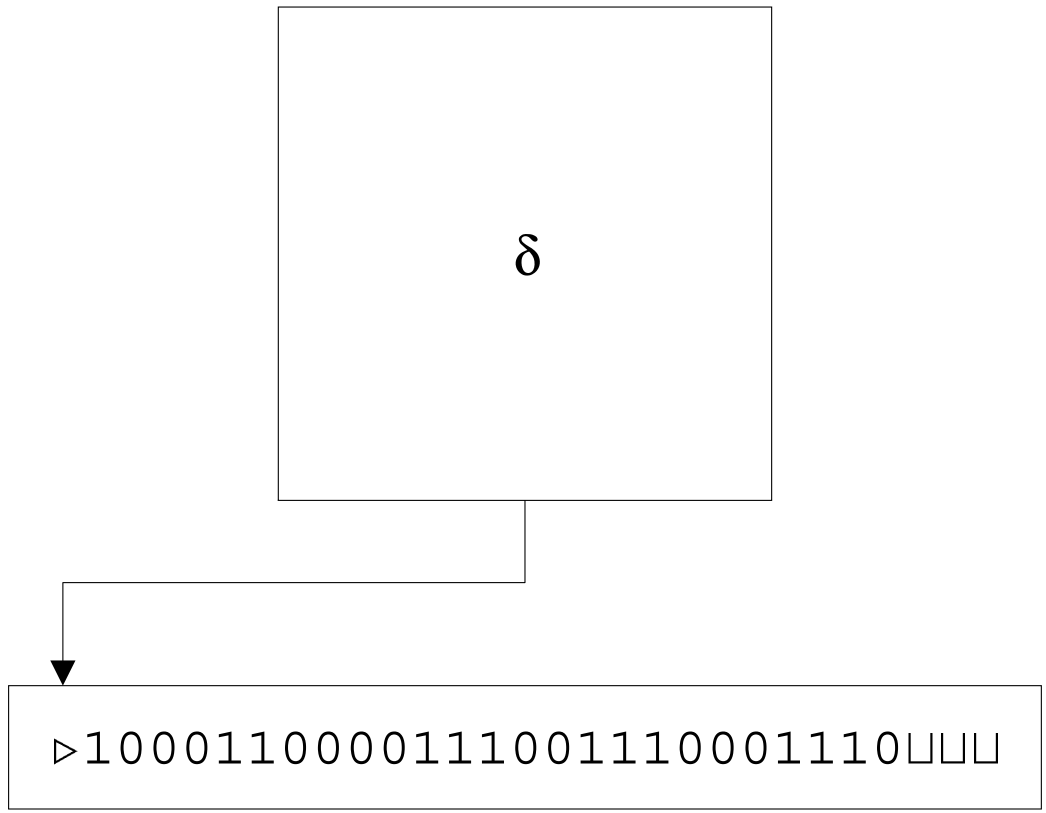
[a] Turing (1936).

[b] Turing (1936), “If we admitted an infinity of states of mind, some of them will be ‘arbitrarily close’ and will be confused.”

A TM Schema



More about δ

- The program has the **halting state** (h), the **accepting state** ("yes"), and the **rejecting state** ("no").
- Given current state $q \in K$ and current symbol $\sigma \in \Sigma$,

$$\delta(q, \sigma) = (p, \rho, D).$$

  - It specifies:
    * The next state p;
    * The symbol ρ to be written over σ;
    * The direction D the cursor will move *afterwards*.
- Assume $\delta(q, \triangleright) = (p, \triangleright, \rightarrow)$.
  - So the cursor never falls off the left end of the string.

More about δ (concluded)

- Think of the program as lines of codes:

$$\begin{aligned}
\delta(q_1, \sigma_1) &= (p_1, \rho_1, D_1), \\
\delta(q_2, \sigma_2) &= (p_2, \rho_2, D_2), \\
&\vdots \\
\delta(q_n, \sigma_n) &= (p_n, \rho_n, D_n).
\end{aligned}$$

- Assume the state is q and the symbol under the cursor σ.
- The line of code that matches (q, σ) is executed.[a]
- Then the process is repeated.

[a]So there should be one and only one instruction for every possible pair (q, σ). Contributed by Mr. Ya-Hsun Chang (`B96902025`, `R00922044`) on September 13, 2011.

The Operations of TMs

- Initially the state is s.
- The string on the tape is initialized to a $\triangleright$, followed by a *finite-length* string $x \in (\Sigma - \{\sqcup\})^*$.
- x is the **input** of the TM.
  - The input must not contain $\sqcup$s (why?)!
- The cursor is pointing to the first symbol, always a $\triangleright$.
- The TM takes each step according to δ.
- The cursor may overwrite $\sqcup$ to make the string longer during the computation.

"Physical" Interpretations

- The tape: computer memory and registers.
  - Except that the tape can be lengthened on demand.
- δ: program.
  - A program has a *finite* size.
- K: instruction numbers.
- s: "`main()`" in the C programming language.
- Σ: **alphabet**, much like the ASCII code.

The Halting of a TM

- A TM M may **halt** in three cases.

  "yes": M **accepts** its input x, and $M(x) =$ "yes".

  "no": M **rejects** its input x, and $M(x) =$ "no".

  h**:** $M(x) = y$ means the string (tape) consists of a $\rhd$, followed by the finite string y which contains no $\sqcup$s, followed by a $\sqcup$.

  - y is the **output** of the computation.
  - y may be empty denoted by ϵ.

- If M never halts on x, then write $M(x) = \nearrow$.

The First TM Program[a]

- Assume $M = (K, \Sigma, \delta, s)$, where $K = \{s, h\}$, $\Sigma = \{\, 0, 1, \sqcup, \triangleright \,\}$, and

$p \in K$	$\sigma \in \Sigma$	$\delta(p, \sigma)$
s	$\triangleright$	$(s, \triangleright, \rightarrow)$
s	1	$(s, 0, \rightarrow)$
s	0	$(s, 1, \rightarrow)$
s	$\sqcup$	$(h, \sqcup, -)$

- This TM converts all 1's in the input string to 0's and vice versa.

[a]Contributed by Mr. Zheyuan (Jeffrey) Gao (`R01922142`) on September 21, 2013.

The Second TM Program[a]

- Assume $M = (K, \Sigma, \delta, s)$, where $K = \{s, s_1, h\}$, $\Sigma = \{\,0, 1, \sqcup, \triangleright\,\}$, and

[a] Contributed by Mr. Zheyuan (Jeffrey) Gao (`R01922142`) on September 21, 2013.

$p \in K$	$\sigma \in \Sigma$	$\delta(p, \sigma)$
s	$\triangleright$	$(s, \triangleright, \rightarrow)$
s	0	$(s, 0, \rightarrow)$
s	1	$(s_1, 1, \rightarrow)$
s_1	0	$(s, 0, \rightarrow)$
s_1	1	$(h, 1, -)$
s	$\sqcup$	$(h, \sqcup, -)$
s_1	$\sqcup$	$(h, \sqcup, -)$

The Second TM Program (concluded)

- This TM scans to the right until it finds two consecutive 1's and then halts.
- Otherwise, it halts at the end of the input string.

The Third TM Program

- Assume $M = (K, \Sigma, \delta, s)$, where $K = \{s, s_1, \text{“yes”}, \text{“no”}\}$, $\Sigma = \{\, 0, 1, \sqcup, \triangleright \,\}$, and

$p \in K$	$\sigma \in \Sigma$	$\delta(p, \sigma)$
s	$\triangleright$	$(s, \triangleright, \rightarrow)$
s	0	$(s, 0, \rightarrow)$
s	1	$(s_1, 1, \rightarrow)$
s_1	0	$(s, 0, \rightarrow)$
s_1	1	$(\text{“yes”}, 1, -)$
s	$\sqcup$	$(\text{“no”}, \sqcup, -)$
s_1	$\sqcup$	$(\text{“no”}, \sqcup, -)$

The Third TM Program (concluded)

- This TM accepts the input if there are two consecutive 1's.
- Otherwise, it rejects the input string.

Why Turing Machines?

- Because of the simplicity of the TM, the model has the advantage when it comes to complexity issues.
- One can conceivably develop a complexity theory based on something similar to C, C++ or Java.
- But the added complexity does not yield additional fundamental insights.
- We will describe TMs in pseudocode only.[a]

[a]But you are strongly encouraged to read and understand the TM codes in the textbook to gain insight on this programming language.

A TM Program To Insert a Symbol

- We want to compute $f(x) = ax$.
  - The TM moves its cursor to the last symbol.
  - It moves the last symbol of x to the right by one position.
  - It moves the next to last symbol to the right, and so on.
  - The TM finally writes a in the first position.
- The total number of steps is $O(n)$, where n is the length of x.

Remarks

- A computation model should be "physically" realizable.
  - E.g., our brain, at least as powerful as a Turing machine, is physical.
- Although a TM requires a tape of potentially infinite length, which is not realizable, it is not a major *conceptual* issue.[a]
  - Imagine you ("the program") are living next to a paper mill while carrying out a TM code using pencil ("the cursor") and paper ("the tape").
  - The mill will produce extra paper if needed.

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[a] Thanks to a lively discussion on September 20, 2006.

Remarks (concluded)

- Even our computer is only an approximation of a TM for the same reason.
  - But it is easy to imagine our computer with more and more address space, memory space, and disk space.

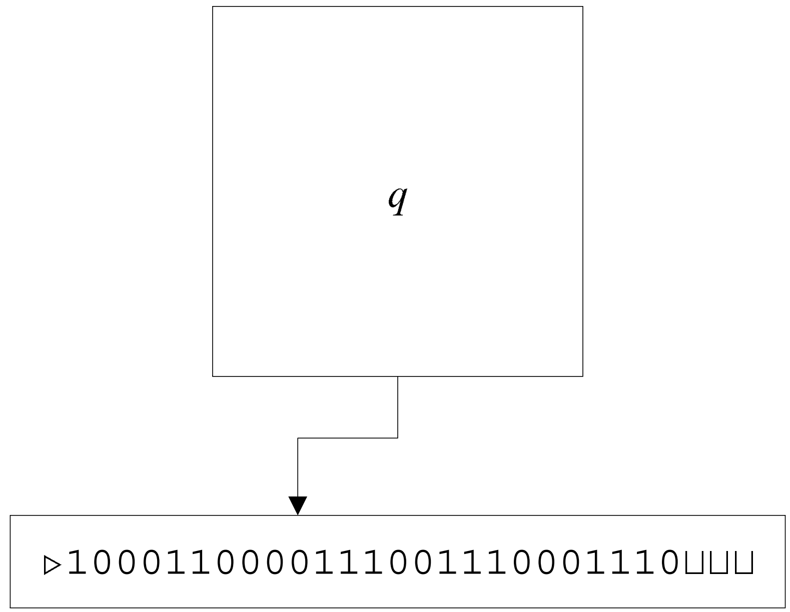
The Concept of Configuration

- A **configuration**[a] is a complete description of the current state of the computation.
- The specification of a configuration is sufficient for the computation to continue as if it had not been stopped.
  - What does your PC save before it sleeps or hibernates?
  - Enough for it to resume work later.
- Similar to the concept of state in Markov processes.

[a]This term was due to Turing (1936).

Configurations (concluded)

- A configuration is a triple (q, w, u):
  - $q \in K$.
  - $w \in \Sigma^*$ is the string to the left of the cursor (inclusive).
  - $u \in \Sigma^*$ is the string to the right of the cursor.
- Note that (w, u) describes both the string and the cursor position.



- $w = \triangleright$1000110000.
- $u =$ 111001110001110.

Yielding

- Fix a TM M.
- Configuration (q, w, u) **yields** configuration (q', w', u') in one step,
$$(q, w, u) \xrightarrow{M} (q', w', u'),$$
if a step of M from configuration (q, w, u) results in configuration (q', w', u').
- $(q, w, u) \xrightarrow{M^k} (q', w', u')$: Configuration (q, w, u) yields configuration (q', w', u') after $k \in \mathbb{N}$ steps.
- $(q, w, u) \xrightarrow{M^*} (q', w', u')$: Configuration (q, w, u) yields configuration (q', w', u').

Alan Turing (1912–1954)

Richard Dawkins (2006), "Turing arguably made a greater contribution to defeating the Nazis than Eisenhower or Churchill."

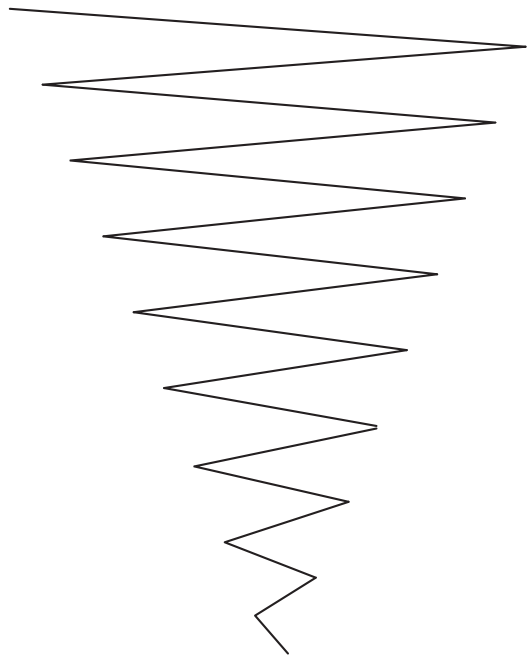
Michael Peck (2014), "But ULTRA didn't detect German preparations, which was taken as an indication that nothing was happening."

Palindromes[a]

- A string is a **palindrome** if it reads the same forwards and backwards (e.g., `001100`).
- A TM program can be written to recognize palindromes:
  - It matches the first character with the last character.
  - It matches the second character with the next to last character, etc. (see next page).
  - "yes" for palindromes and "no" for nonpalindromes.
- This program takes $O(n^2)$ steps.
- Can we do better?

[a]Bryson (2001), "Possibly the most demanding form of wordplay in English[.]"

100011000000100111



A Matching Lower Bound for PALINDROME

Theorem 1 (Hennie (1965)) PALINDROME *on single-string TMs takes* $\Omega(n^2)$ *steps in the worst case.*

Comments on Lower-Bound Proofs

- They are usually difficult.
  - Worthy of a Ph.D. degree.
- An algorithm whose running time matches a lower bound means it is optimal.
  - The simple $O(n^2)$ algorithm for PALINDROME is optimal.
- This happens rarely and is model dependent.
  - Searching, sorting, PALINDROME, matrix-vector multiplication, etc.

The Kleene Star Operation $*$[a]

- Let A be a set.
- The **Kleene star** of A, denoted by A^*, is the set of all strings obtained by concatenating zero or more strings from A.
  - For example, suppose $A = \{0, 1\}$.
  - Then
  $$A^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}.$$
  - Note that every string in A^* must be of finite length.

[a] Kleene (1956).

Decidability and Recursive Languages

- Let $L \subseteq (\Sigma - \{\sqcup\})^*$ be a **language**, i.e., a set of strings of non-$\sqcup$ symbols, with a *finite* length.
  - For example, $\{0, 01, 10, 210, 1010, \ldots\}$.
- Let M be a TM such that for any string x:
  - If $x \in L$, then $M(x) =$ "yes."
  - If $x \notin L$, then $M(x) =$ "no."
- We say M **decides** L.
- If there exists a TM that decides L, then L is **recursive**[a] or **decidable**.

[a] Little to do with the concept of "recursive" calls.

Recursive and Nonrecursive Languages: Examples

- The set of palindromes over any alphabet is recursive.[a]
  - PALINDROME cannot be solved by finite state automata.
  - In fact, finite state automata are equivalent to read-only, right-moving Turing machines.[b]
- The set of prime numbers $\{2, 3, 5, 7, 11, 13, 17, \ldots\}$ is recursive.[c]

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[a] Need a program that returns "yes" iff the input is a palindrome.
[b] Thanks to a lively discussion on September 15, 2015.
[c] Need a program that returns "yes" iff the input is a prime.

Recursive and Nonrecursive Languages: Examples (concluded)

- The set of C programs that do not contain a `while`, a `for`, or a `goto` is recursive.[a]
- But, the set of C programs that do not contain an infinite loop is *not* recursive (see p. 148).

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[a]Need a program that returns "yes" iff the input C code does not contain any of the keywords.