KNAPSACK Is NP-Complete[a]

- KNAPSACK $\in$ NP: Guess an S and check the constraints.
- We shall reduce EXACT COVER BY 3-SETS to KNAPSACK, in which $v_i = w_i$ for all i and $K = W$.
- The simplified KNAPSACK now asks if a subset of $v_1, v_2, \ldots, v_n$ adds up to exactly K.[b]
  - Picture yourself as a radio DJ.

---

[a] Karp (1972).

[b] This problem is called SUBSET SUM.

The Proof (continued)

- The primary differences between the two problems are:[a]
  - Sets vs. numbers.
  - Union vs. addition.
- We are given a family $F = \{S_1, S_2, \ldots, S_n\}$ of size-3 subsets of $U = \{1, 2, \ldots, 3m\}$.
- EXACT COVER BY 3-SETS asks if there are m disjoint sets in F that cover the set U.

[a] Thanks to a lively class discussion on November 16, 2010.

The Proof (continued)

- Think of a set as a bit vector in $\{0,1\}^{3m}$.
  - Assume $m = 3$.
  - `110010000` means the set $\{1, 2, 5\}$.
  - `001100010` means the set $\{3, 4, 8\}$.
- Assume there are $n = 5$ size-3 subsets in F.
- Our goal is

$$\overbrace{\texttt{1}\,\texttt{1}\cdots\texttt{1}}^{3m}.$$

The Proof (continued)

- A bit vector can also be seen as a binary *number*.
- Set union resembles addition:

$$\begin{array}{rr} & \texttt{001100010} \\ + & \texttt{110010000} \\ \hline & \texttt{111110010} \end{array}$$

which denotes the set $\{1, 2, 3, 4, 5, 8\}$, as desired.

The Proof (continued)

- Trouble occurs when there is *carry*:

$$\begin{array}{rr} & \texttt{010000000} \\ + & \texttt{010000000} \\ \hline & \texttt{100000000} \end{array}$$

  which denotes the set $\{1\}$, not the desired $\{2\}$.

The Proof (continued)

- Or consider

$$\begin{array}{rr} & \texttt{001100010} \\ + & \texttt{001110000} \\ \hline & \texttt{011010010} \end{array}$$

which denotes the set $\{2, 3, 5, 8\}$, not the desired $\{3, 4, 5, 8\}$.[a]

[a]Corrected by Mr. Chihwei Lin (D97922003) on January 21, 2010.

The Proof (continued)

- Carry may also lead to a situation where we obtain our solution $\texttt{1}\,\texttt{1}\cdots\texttt{1}$ with more than m sets in F.
- For example,

$$\begin{array}{rr} & \texttt{000100010} \\ & \texttt{001110000} \\ & \texttt{101100000} \\ + & \texttt{000001101} \\ \hline & \texttt{111111111} \end{array}$$

- But the correct answer, $\{1, 3, 4, 5, 6, 7, 8, 9\}$, is *not* an exact cover.

The Proof (continued)

- And it uses 4 sets instead of the required $m = 3$.[a]
- To fix this problem, we enlarge the base just enough so that there are no carries.[b]
- Because there are n vectors in total, we change the base from 2 to $n + 1$.

---

[a] Thanks to a lively class discussion on November 20, 2002.

[b] You cannot map $\cup$ to $\vee$ because KNAPSACK requires $+$.

The Proof (continued)

- Set v_i to be the integer corresponding to the bit vector encoding S_i in base $n+1$:

$$v_i = \sum_{j \in S_i} 1 \times (n+1)^{3m-j} \tag{4}$$

- Set

$$K = \sum_{j=0}^{3m-1} 1 \times (n+1)^j = \overbrace{1\,1 \cdots 1}^{3m} \quad \text{(base } n+1\text{)}.$$

- Now in base $n+1$, if there is a set S such that $\sum_{i \in S} v_i = \overbrace{1\,1 \cdots 1}^{3m}$, then every position must be contributed by exactly one v_i and $|S| = m$.

The Proof (continued)

- For example, the case on p. 420 becomes

$$
\begin{array}{rr}
 & \texttt{000100010} \\
 & \texttt{001110000} \\
 & \texttt{101100000} \\
+ & \texttt{000001101} \\
\hline
 & \texttt{102311111}
\end{array}
$$

  in base $n + 1 = 6$.

- It does not meet the goal.

The Proof (continued)

- Suppose F admits an exact cover, say $\{S_1, S_2, \ldots, S_m\}$.
- Then picking $S = \{1, 2, \ldots, m\}$ clearly results in

$$v_1 + v_2 + \cdots + v_m = \overbrace{1\,1 \cdots 1}^{3m}.$$

- It is important to note that the meaning of addition (+) is independent of the base.[a]
  - It is just regular addition.
  - But an S_i may give rise to different integer v_i's in Eq. (4) on p. 422 under different bases.

[a]Contributed by Mr. Kuan-Yu Chen (R92922047) on November 3, 2004.

The Proof (concluded)

- On the other hand, suppose there exists an S such that

$$\sum_{i \in S} v_i = \overbrace{1\,1 \cdots 1}^{3m}$$

in base $n+1$.

- The no-carry property implies that $|S| = m$ and

$$\{S_i : i \in S\}$$

is an exact cover.

An Example

- Let $m = 3$, $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and

$$\begin{aligned} S_1 &= \{1, 3, 4\}, \\ S_2 &= \{2, 3, 4\}, \\ S_3 &= \{2, 5, 6\}, \\ S_4 &= \{6, 7, 8\}, \\ S_5 &= \{7, 8, 9\}. \end{aligned}$$

- Note that $n = 5$, as there are 5 S_i's.

An Example (continued)

- Our reduction produces

$$
\begin{aligned}
K &= \sum_{j=0}^{3\times 3-1} 6^j = \overbrace{1\,1\cdots 1}^{3\times 3} \quad \text{(base 6)} = 2015539,\\
v_1 &= \texttt{101100000} = 1734048,\\
v_2 &= \texttt{011100000} = 334368,\\
v_3 &= \texttt{010011000} = 281448,\\
v_4 &= \texttt{000001110} = 258,\\
v_5 &= \texttt{000000111} = 43.
\end{aligned}
$$

An Example (concluded)

- Note $v_1 + v_3 + v_5 = K$ because

$$\begin{array}{rr} & \texttt{101100000} \\ & \texttt{010011000} \\ + & \texttt{000000111} \\ \hline & \texttt{111111111} \end{array}$$

- Indeed,

$$S_1 \cup S_3 \cup S_5 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\},$$

an exact cover by 3-sets.

BIN PACKING

- We are given N positive integers $a_1, a_2, \ldots, a_N$, an integer C (the capacity), and an integer B (the number of bins).
- BIN PACKING asks if these numbers can be partitioned into B subsets, each of which has total sum at most C.
- Think of packing bags at the check-out counter.

Theorem 47 BIN PACKING *is NP-complete.*

BIN PACKING (concluded)

- But suppose $a_1, a_2, \ldots, a_N$ are randomly distributed between 0 and 1.
- Let B be the smallest number of unit-capacity bins capable of holding them.
- Then B can differ from its average by more than t with probability at most $2e^{-2t^2/N}$.[a]

[a] Dubhashi and Panconesi (2012).

INTEGER PROGRAMMING

- INTEGER PROGRAMMING asks whether a system of linear inequalities with integer coefficients has an integer solution.
- In contrast, LINEAR PROGRAMMING asks whether a system of linear inequalities with integer coefficients has a *rational* solution.

INTEGER PROGRAMMING Is NP-Complete[a]

- SET COVERING can be expressed by the inequalities $Ax \geq \vec{1}$, $\sum_{i=1}^{n} x_i \leq B$, $0 \leq x_i \leq 1$, where
  - x_i is one if and only if S_i is in the cover.
  - A is the matrix whose columns are the bit vectors of the sets $S_1, S_2, \ldots$.
  - $\vec{1}$ is the vector of 1s.
  - The operations in Ax are standard matrix operations.
- This shows INTEGER PROGRAMMING is NP-hard.
- Many NP-complete problems can be expressed as an INTEGER PROGRAMMING problem.

[a] Karp (1972).

Easier or Harder?[a]

- Adding restrictions on the allowable *problem instances* will not make a problem harder.
  - We are now solving a subset of problem instances or special cases.
  - The INDEPENDENT SET proof (p. 361) and the KNAPSACK proof (p. 414).
  - SAT to 2SAT (easier by p. 342).
  - CIRCUIT VALUE to MONOTONE CIRCUIT VALUE (equally hard by p. 314).

[a]Thanks to a lively class discussion on October 29, 2003.

Easier or Harder? (concluded)

- Adding restrictions on the allowable *solutions* (the solution space) may make a problem harder, equally hard, or easier.
- It is problem dependent.
  - MIN CUT to BISECTION WIDTH (harder by p. 389).
  - LINEAR PROGRAMMING to INTEGER PROGRAMMING (harder by p. 431).
  - SAT to NAESAT (equally hard by p. 355) and MAX CUT to MAX BISECTION (equally hard by p. 387).
  - 3-COLORING to 2-COLORING (easier by p. 398).

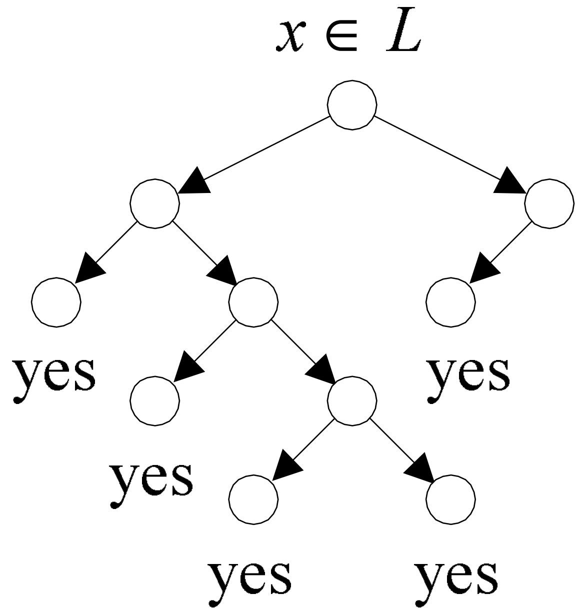
coNP and Function Problems

coNP

- NP is the class of problems that have succinct certificates (recall Proposition 36 on p. 326).
- By definition, coNP is the class of problems whose complement is in NP.
- coNP is therefore the class of problems that have succinct disqualifications:
  - A "no" instance of a problem in coNP possesses a short proof of its being a "no" instance.
  - Only "no" instances have such proofs.

coNP (continued)

- Suppose L is a coNP problem.
- There exists a polynomial-time nondeterministic algorithm M such that:
  - If $x \in L$, then $M(x) =$ "yes" for all computation paths.
  - If $x \notin L$, then $M(x) =$ "no" for some computation path.
- Note that if we swap "yes" and "no" of M, the new algorithm M' decides $\bar{L} \in$ NP in the classic sense (p. 109).


$x \in L$
yes
yes
yes
yes
yes
$x \notin L$
no
yes
yes
no
yes

coNP (continued)

- So there are 3 major approaches to proving $L \in$ coNP.
  1. Prove $\bar{L} \in$ NP.
  2. Prove that "no" instances possess short proofs.
  3. Write an algorithm for it.

coNP (concluded)

- Clearly $\mathrm{P} \subseteq \mathrm{coNP}$.
- It is not known if

$$\mathrm{P} = \mathrm{NP} \cap \mathrm{coNP}.$$

  – Contrast this with

$$\mathrm{R} = \mathrm{RE} \cap \mathrm{coRE}$$

  (see Proposition 12 on p. 170).

Some coNP Problems

- VALIDITY $\in$ coNP.
  - If ϕ is not valid, it can be disqualified very succinctly: a truth assignment that does not satisfy it.
- SAT COMPLEMENT $\in$ coNP.
  - SAT COMPLEMENT is the complement of SAT.
  - The disqualification is a truth assignment that satisfies it.
- HAMILTONIAN PATH COMPLEMENT $\in$ coNP.
  - The disqualification is a Hamiltonian path.

Some coNP Problems (concluded)

- OPTIMAL TSP (D) $\in$ coNP.
  - OPTIMAL TSP (D) asks if the optimal tour has a total distance of B, where B is an input.[a]
  - The disqualification is a tour with a length $< B$.

---

[a] Defined by Mr. Che-Wei Chang (R95922093) on September 27, 2006.

A Nondeterministic Algorithm for SAT COMPLEMENT

ϕ is a boolean formula with n variables.

for $i = 1, 2, \ldots, n$ **do**
    Guess $x_i \in \{0, 1\}$; {Nondeterministic choice.}
end for
{Verification:}
if $\phi(x_1, x_2, \ldots, x_n) = 1$ **then**
    "no";
else
    "yes";
end if

Analysis

- The algorithm decides language $\{\phi : \phi \text{ is unsatisfiable}\}$.
  - The computation tree is a complete binary tree of depth n.
  - Every computation path corresponds to a particular truth assignment out of 2^n.
  - ϕ is unsatisfiable iff every truth assignment falsifies ϕ.
  - But every truth assignment falsifies ϕ iff every computation path results in "yes."

An Alternative Characterization of coNP

Proposition 48 *Let $L \subseteq \Sigma^*$ be a language. Then $L \in coNP$ if and only if there is a polynomially decidable and polynomially balanced relation R such that*

$$L = \{x : \forall y\, (x, y) \in R\}.$$

(As on p. 325, we assume $|\,y\,| \leq |\,x\,|^k$ for some k.)

- $\bar{L} = \{x : \exists y\, (x, y) \in \neg R\}$.
- Because $\neg R$ remains polynomially balanced, $\bar{L} \in$ NP by Proposition 36 (p. 326).
- Hence $L \in$ coNP by definition.

coNP-Completeness

Proposition 49 *L is NP-complete if and only if its complement $\bar{L} = \Sigma^* - L$ is coNP-complete.*

Proof ($\Rightarrow$; the $\Leftarrow$ part is symmetric)

- Let $\bar{L}'$ be any coNP language.
- Hence $L' \in$ NP.
- Let R be the reduction from L' to L.
- So $x \in L'$ if and only if $R(x) \in L$.
- Equivalently, $x \notin L'$ if and only if $R(x) \notin L$ (the law of transposition).

coNP Completeness (concluded)

- So $x \in \bar{L}'$ if and only if $R(x) \in \bar{L}$.
- R is a reduction from $\bar{L}'$ to $\bar{L}$.
- This shows $\bar{L}$ is coNP-hard.
- But $\bar{L} \in \text{coNP}$.
- This shows $\bar{L}$ is coNP-complete.

Some coNP-Complete Problems

- SAT COMPLEMENT is coNP-complete.
- VALIDITY is coNP-complete.
  - ϕ is valid if and only if $\neg\phi$ is not satisfiable.
  - The reduction from SAT COMPLEMENT to VALIDITY is hence easy.
- HAMILTONIAN PATH COMPLEMENT is coNP-complete.

Possible Relations between P, NP, coNP

1. P = NP = coNP.
2. NP = coNP but P $\neq$ NP.
3. NP $\neq$ coNP and P $\neq$ NP.
   - This is the current "consensus."[a]

[a] Carl Gauss (1777–1855), "I could easily lay down a multitude of such propositions, which one could neither prove nor dispose of."

The Primality Problem

- An integer p is **prime** if $p > 1$ and all positive numbers other than 1 and p itself cannot divide it.
- PRIMES asks if an integer N is a prime number.
- Dividing N by $2, 3, \ldots, \sqrt{N}$ is *not* efficient.
  - The length of N is only $\log N$, but $\sqrt{N} = 2^{0.5 \log N}$.
  - So it is an exponential-time algorithm.
- A polynomial-time algorithm for PRIMES was not found until 2002 by Agrawal, Kayal, and Saxena!
- Later, we will focus on efficient "probabilistic" algorithms for PRIMES (used in *Mathematica*, e.g.).

if $n = a^b$ for some $a, b > 1$ **then**

  return "composite";

end if

for $r = 2, 3, \ldots, n-1$ **do**

  if $\gcd(n, r) > 1$ **then**

    return "composite";

  end if

  if r is a prime **then**

    Let q be the largest prime factor of $r - 1$;

    if $q \geq 4\sqrt{r}\log n$ and $n^{(r-1)/q} \neq 1 \bmod r$ **then**

      break; {Exit the for-loop.}

    end if

  end if

end for{$r - 1$ has a prime factor $q \geq 4\sqrt{r}\log n$.}

for $a = 1, 2, \ldots, 2\sqrt{r}\log n$ **do**

  if $(x-a)^n \neq (x^n - a) \bmod (x^r - 1)$ in $Z_n[\,x\,]$ **then**

    return "composite";

  end if

end for

return "prime"; {The only place with "prime" output.}

The Primality Problem (concluded)

- $\mathrm{NP} \cap \mathrm{coNP}$ is the class of problems that have succinct certificates and succinct disqualifications.
  - Each "yes" instance has a succinct certificate.
  - Each "no" instance has a succinct disqualification.
  - No instances have both.
- We will see that PRIMES $\in \mathrm{NP} \cap \mathrm{coNP}$.
  - In fact, PRIMES $\in$ P as mentioned earlier.

Primitive Roots in Finite Fields

Theorem 50 (Lucas and Lehmer (1927)) [a] *A number $p > 1$ is a prime if and only if there is a number $1 < r < p$ such that*

1. $r^{p-1} = 1 \bmod p$, *and*
2. $r^{(p-1)/q} \neq 1 \bmod p$ *for all prime divisors q of $p - 1$.*

- This r is called the **primitive root** or **generator**.
- We will prove the theorem later (see pp. 464ff).

[a] François Edouard Anatole Lucas (1842–1891); Derrick Henry Lehmer (1905–1991).

Derrick Lehmer (1905–1991)

Pratt's Theorem

Theorem 51 (Pratt (1975)) PRIMES $\in NP \cap coNP$.

- PRIMES is in coNP because a succinct disqualification is a proper divisor.
  - A proper divisor of a number n means n is *not* a prime.
- Now suppose p is a prime.
- p's certificate includes the r in Theorem 50 (p. 453).
- Use recursive doubling to check if $r^{p-1} = 1 \bmod p$ in time polynomial in the length of the input, $\log_2 p$.
  - $r, r^2, r^4, \ldots \bmod p$, a total of $\sim \log_2 p$ steps.

The Proof (concluded)

- We also need all *prime* divisors of $p-1$: $q_1, q_2, \ldots, q_k$.
  - Whether $r, q_1, \ldots, q_k$ are easy to find is irrelevant.
  - There may be multiple choices for r.
- Checking $r^{(p-1)/q_i} \neq 1 \bmod p$ is also easy.
- Checking $q_1, q_2, \ldots, q_k$ are all the divisors of $p-1$ is easy.
- We still need certificates for the primality of the q_i's.
- The complete certificate is recursive and tree-like:

$$C(p) = (r; q_1, C(q_1), q_2, C(q_2), \ldots, q_k, C(q_k)).$$

- We next prove that $C(p)$ is succinct.
- As a result, $C(p)$ can be checked in polynomial time.

The Succinctness of the Certificate

Lemma 52 *The length of $C(p)$ is at most quadratic at $5 \log_2^2 p$.*

- This claim holds when $p = 2$ or $p = 3$.
- In general, $p - 1$ has $k \leq \log_2 p$ prime divisors $q_1 = 2, q_2, \ldots, q_k$.
  - Reason:

$$2^k \leq \prod_{i=1}^{k} q_i \leq p - 1.$$

- Note also that, as $q_1 = 2$,

$$\prod_{i=2}^{k} q_i \leq \frac{p-1}{2}. \tag{5}$$

The Proof (continued)

- $C(p)$ requires:
  - 2 parentheses;
  - $2k < 2\log_2 p$ separators (at most $2\log_2 p$ bits);
  - r (at most $\log_2 p$ bits);
  - $q_1 = 2$ and its certificate 1 (at most 5 bits);
  - $q_2, \ldots, q_k$ (at most $2\log_2 p$ bits);[a]
  - $C(q_2), \ldots, C(q_k)$.

[a] Why?

The Proof (concluded)

- $C(p)$ is succinct because, by induction,

$$
\begin{aligned}
|C(p)| &\leq 5 \log_2 p + 5 + 5 \sum_{i=2}^{k} \log_2^2 q_i \\
&\leq 5 \log_2 p + 5 + 5 \left(\sum_{i=2}^{k} \log_2 q_i \right)^2 \\
&\leq 5 \log_2 p + 5 + 5 \log_2^2 \frac{p-1}{2} \quad \text{by inequality (5)} \\
&< 5 \log_2 p + 5 + 5(\log_2 p - 1)^2 \\
&= 5 \log_2^2 p + 10 - 5 \log_2 p \leq 5 \log_2^2 p
\end{aligned}
$$

  for $p \geq 4$.

A Certificate for 23[a]

- Note that 7 is a primitive root modulo 23 and $23 - 1 = 22 = 2 \times 11$.
- So
$$C(23) = (7; 2, C(2), 11, C(11)).$$
- Note that 2 is a primitive root modulo 11 and $11 - 1 = 10 = 2 \times 5$.
- So
$$C(11) = (2; 2, C(2), 5, C(5)).$$

[a]Thanks to a lively discussion on April 24, 2008.

A Certificate for 23 (concluded)

- Note that 2 is a primitive root modulo 5 and $5 - 1 = 4 = 2^2$.
- So
$$C(5) = (2; 2, C(2)).$$
- In summary,
$$C(23) = (7; 2, C(2), 11, (2; 2, C(2), 5, (2; 2, C(2)))).$$

Basic Modular Arithmetics[a]

- Let $m, n \in \mathbb{Z}^+$.
- $m \mid n$ means m divides n; m is n's **divisor**.
- We call the numbers $0, 1, \ldots, n-1$ the **residue** modulo n.
- The **greatest common divisor** of m and n is denoted $\gcd(m, n)$.
- The r in Theorem 50 (p. 453) is a primitive root of p.
- We now prove the existence of primitive roots and then Theorem 50 (p. 453).

[a] Carl Friedrich Gauss.

Basic Modular Arithmetics (concluded)

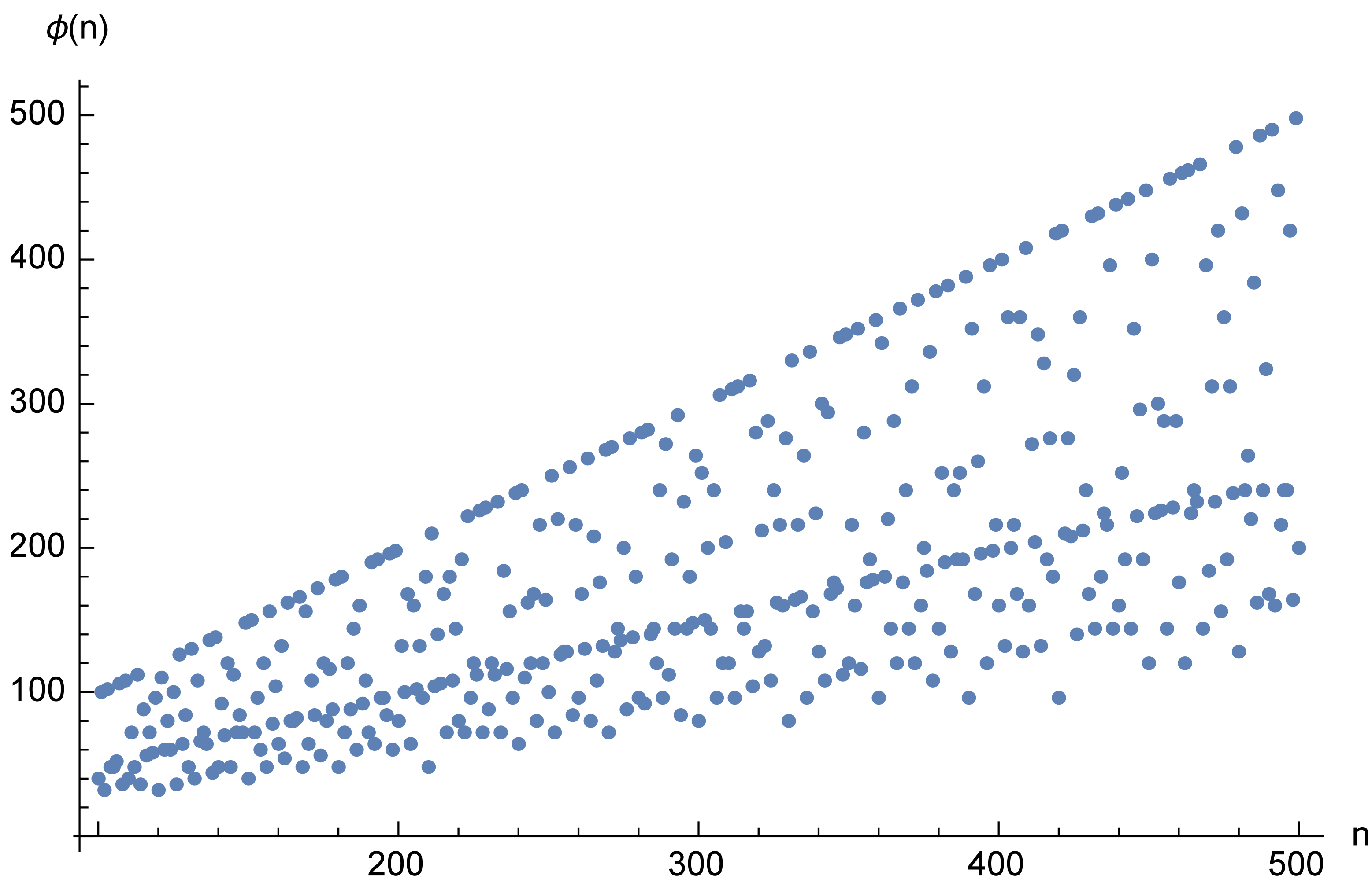
- We use
$$a \equiv b \mod n$$
if $n \mid (a - b)$.
  - So $25 \equiv 38 \bmod 13$.
- We use
$$a = b \bmod n$$
if b is the remainder of a divided by n.
  - So $25 = 12 \bmod 13$.

Euler's[a] Totient or Phi Function

- Let
$$\Phi(n) = \{m : 1 \leq m < n, \gcd(m, n) = 1\}$$
be the set of all positive integers less than n that are prime to n.[b]
  - $\Phi(12) = \{1, 5, 7, 11\}$.
- Define **Euler's function** of n to be $\phi(n) = |\Phi(n)|$.
- $\phi(p) = p - 1$ for prime p, and $\phi(1) = 1$ by convention.
- Euler's function is not expected to be easy to compute without knowing n's factorization.

---

[a] Leonhard Euler (1707–1783).
[b] Z_n^* is an alternative notation.


ϕ(n)
500
400
300
200
100
200
300
400
500
n

Two Properties of Euler's Function

The inclusion-exclusion principle[a] can be used to prove the following.

Lemma 53 $\phi(n) = n \prod_{p \mid n} (1 - \frac{1}{p})$.

- If $n = p_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell}$ is the prime factorization of n, then

$$\phi(n) = n \prod_{i=1}^{\ell} \left(1 - \frac{1}{p_i}\right).$$

Corollary 54 $\phi(mn) = \phi(m)\, \phi(n)$ *if* $\gcd(m, n) = 1$.

[a] Consult any textbook on discrete mathematics.

A Key Lemma

Lemma 55 $\sum_{m \mid n} \phi(m) = n$.

- Let $n = \prod_{i=1}^{\ell} p_i^{k_i}$ be the prime factorization of n and consider

$$\prod_{i=1}^{\ell} [\, \phi(1) + \phi(p_i) + \cdots + \phi(p_i^{k_i}) \,]. \tag{6}$$

- Equation (6) equals n because $\phi(p_i^k) = p_i^k - p_i^{k-1}$ by Lemma 53 (p. 466) so $\phi(1) + \phi(p_i) + \cdots + \phi(p_i^{k_i}) = p_i^{k_i}$.
- Expand Eq. (6) to yield

$$n = \sum_{k_1' \leq k_1, \ldots, k_\ell' \leq k_\ell} \prod_{i=1}^{\ell} \phi(p_i^{k_i'}).$$

The Proof (concluded)

- By Corollary 54 (p. 466),

$$\prod_{i=1}^{\ell} \phi(p_i^{k'_i}) = \phi\left(\prod_{i=1}^{\ell} p_i^{k'_i}\right).$$

- So Eq. (6) becomes

$$n = \sum_{k'_1 \leq k_1, \ldots, k'_\ell \leq k_\ell} \phi\left(\prod_{i=1}^{\ell} p_i^{k'_i}\right).$$

- Each $\prod_{i=1}^{\ell} p_i^{k'_i}$ is a unique divisor of $n = \prod_{i=1}^{\ell} p_i^{k_i}$.
- Equation (6) becomes

$$\sum_{m \mid n} \phi(m).$$

Leonhard Euler (1707–1783)