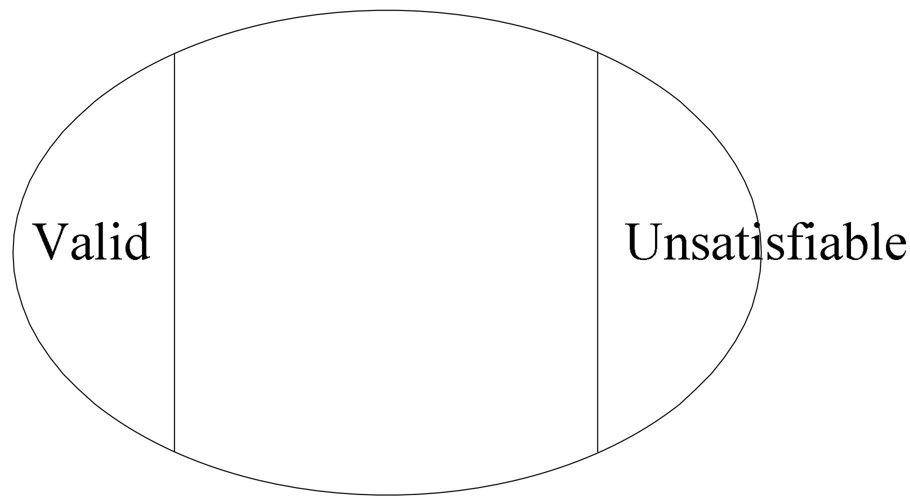
SATISFIABILITY (SAT)

- The **length** of a boolean expression is the length of the string encoding it.
- SATISFIABILITY (SAT): Given a CNF ϕ, is it satisfiable?
- Solvable in exponential time on a TM by the truth table method.
- Solvable in polynomial time on an NTM, hence in NP (p. 119).
- A most important problem in settling the "P $\stackrel{?}{=}$ NP" problem (p. 312).

UNSATISFIABILITY (UNSAT or SAT COMPLEMENT) and VALIDITY

- UNSAT (SAT COMPLEMENT): Given a boolean expression ϕ, is it unsatisfiable?
- VALIDITY: Given a boolean expression ϕ, is it valid?
  - ϕ is valid if and only if $\neg\phi$ is unsatisfiable.
  - ϕ and $\neg\phi$ are basically of the same length.
  - So UNSAT and VALIDITY have the same complexity.
- Both are solvable in exponential time on a TM by the truth table method.
- Can we do better?

Relations among SAT, UNSAT, and VALIDITY



- The negation of an unsatisfiable expression is a valid expression.
- None of the three problems—satisfiability, unsatisfiability, validity—are known to be in P.

Boolean Functions

- An n-ary boolean function is a function

$$f : \{\texttt{true}, \texttt{false}\}^n \to \{\texttt{true}, \texttt{false}\}.$$

- It can be represented by a truth table.
- There are 2^{2^n} such boolean functions.
  - We can assign `true` or `false` to f for each of the 2^n truth assignments.
- How about $\{\texttt{true}, \texttt{false}\}^n \to \{\texttt{true}, \texttt{false}\}^m$?

Boolean Functions (continued)

Assignment	Truth value
1	`true` or `false`
2	`true` or `false`
$\vdots$	$\vdots$
2^n	`true` or `false`

Boolean Functions (continued)

- A boolean expression expresses a boolean function.
  - Think of its truth value under all truth assignments.
- A boolean function expresses a boolean expression.
  - $\bigvee_{T \models \phi, \text{ literal } y_i \text{ is true in "row" } T} (y_1 \wedge \cdots \wedge y_n)$.
    * $y_1 \wedge \cdots \wedge y_n$ is called the **minterm** over $\{x_1, \ldots, x_n\}$ for T.[a]
  - The size[b] is $\leq n2^n \leq 2^{2n}$.

[a] Similar to **programmable logic array**.
[b] We count only the literals here.

Boolean Functions (continued)

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

The corresponding boolean expression:

$$(\neg x_1 \land \neg x_2) \lor (\neg x_1 \land x_2) \lor (x_1 \land x_2).$$

Boolean Functions (concluded)

Corollary 15 *Every n-ary boolean function can be expressed by a boolean expression of size $O(n2^n)$.*

- In general, the exponential length in n cannot be avoided (p. 211).
- The size of the truth table is also $O(n2^n)$.

Boolean Circuits

- A **boolean circuit** is a graph C whose nodes are the **gates**.
- There are no cycles in C.
- All nodes have indegree (number of incoming edges) equal to 0, 1, or 2.
- Each gate has a **sort** from

$$\{\texttt{true}, \texttt{false}, \lor, \land, \neg, x_1, x_2, \ldots\}.$$

  - There are $n + 5$ sorts.

Boolean Circuits (concluded)

- Gates with a sort from $\{\texttt{true}, \texttt{false}, x_1, x_2, \ldots\}$ are the **inputs** of C and have an indegree of zero.
- The **output gate**(s) has no outgoing edges.
- A boolean circuit computes a boolean function.
- The same boolean function can be computed by infinitely many equivalent boolean circuits.

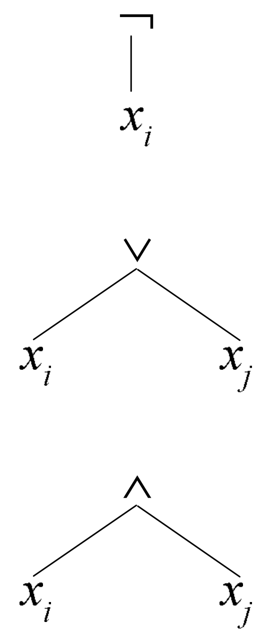
Boolean Circuits and Expressions

- They are equivalent representations.
- One can construct one from the other:

$\neg x_i$

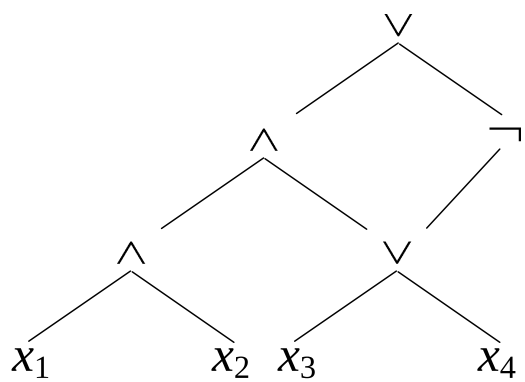
$x_i \vee x_j$

$x_i \wedge x_j$



An Example

$$((x_1 \wedge x_2) \wedge (x_3 \vee x_4)) \vee (\neg (x_3 \vee x_4))$$



- Circuits are more economical because of the possibility of sharing.

CIRCUIT SAT and CIRCUIT VALUE

CIRCUIT SAT**:** Given a circuit, is there a truth assignment such that the circuit outputs true?

- CIRCUIT SAT $\in$ NP: Guess a truth assignment and then evaluate the circuit.

CIRCUIT VALUE**:** The same as CIRCUIT SAT except that the circuit has no variable gates.

- CIRCUIT VALUE $\in$ P: Evaluate the circuit from the input gates gradually towards the output gate.

Some Boolean Functions Need Exponential Circuits[a]

Theorem 16 (Shannon (1949)) *For any $n \geq 2$, there is an n-ary boolean function f such that no boolean circuits with $2^n/(2n)$ or fewer gates can compute it.*

- There are 2^{2^n} different n-ary boolean functions (p. 201).
- So it suffices to prove that the number of boolean circuits with $2^n/(2n)$ or fewer gates is less than 2^{2^n}.

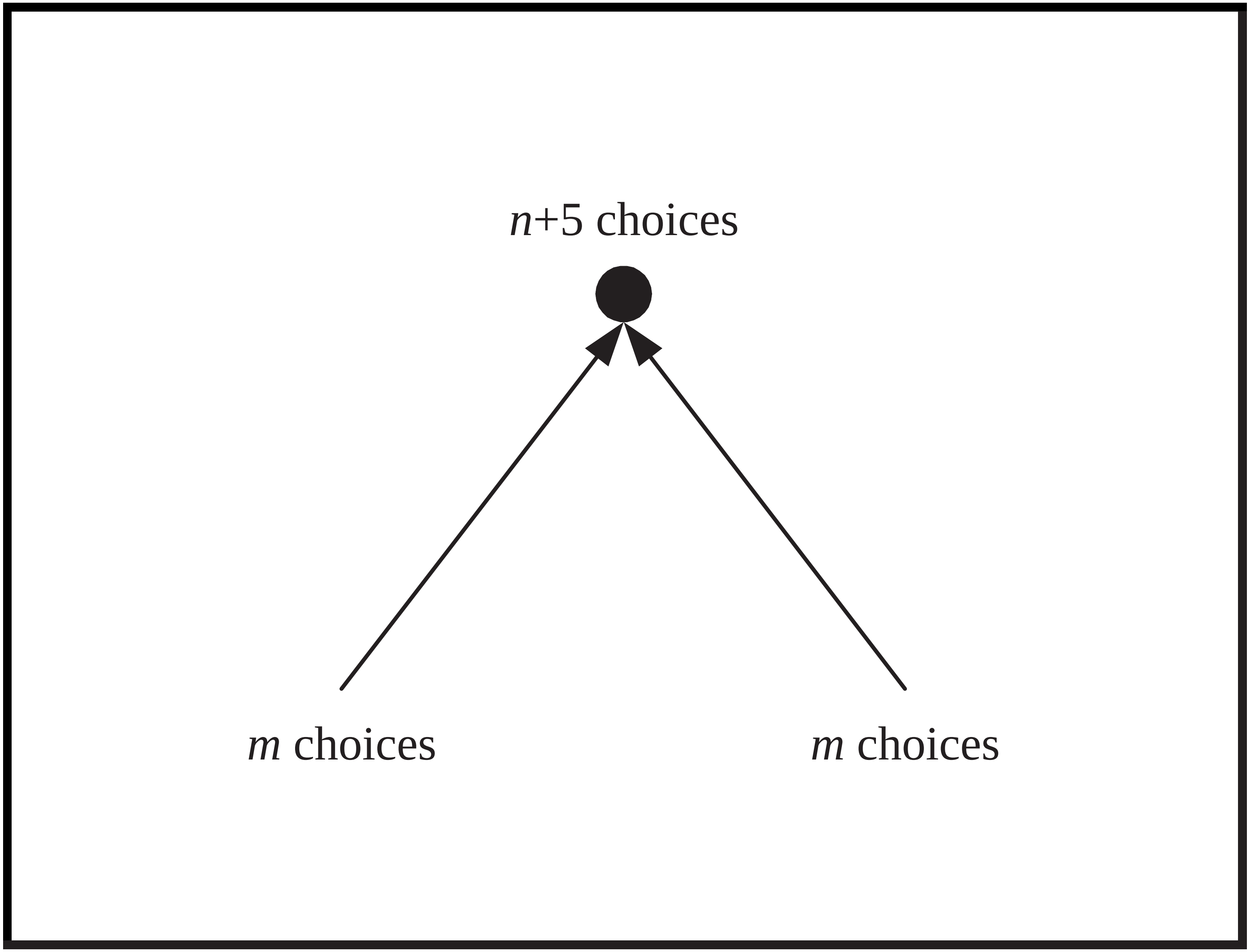
[a] Can be strengthened to "almost all boolean functions ..."

The Proof (concluded)

- There are at most $((n+5) \times m^2)^m$ boolean circuits with m or fewer gates (see next page).
- But $((n+5) \times m^2)^m < 2^{2^n}$ when $m = 2^n/(2n)$:

$$\begin{aligned} & m \log_2((n+5) \times m^2) \\ = \quad & 2^n \left(1 - \frac{\log_2 \frac{4n^2}{n+5}}{2n} \right) \\ < \quad & 2^n \end{aligned}$$

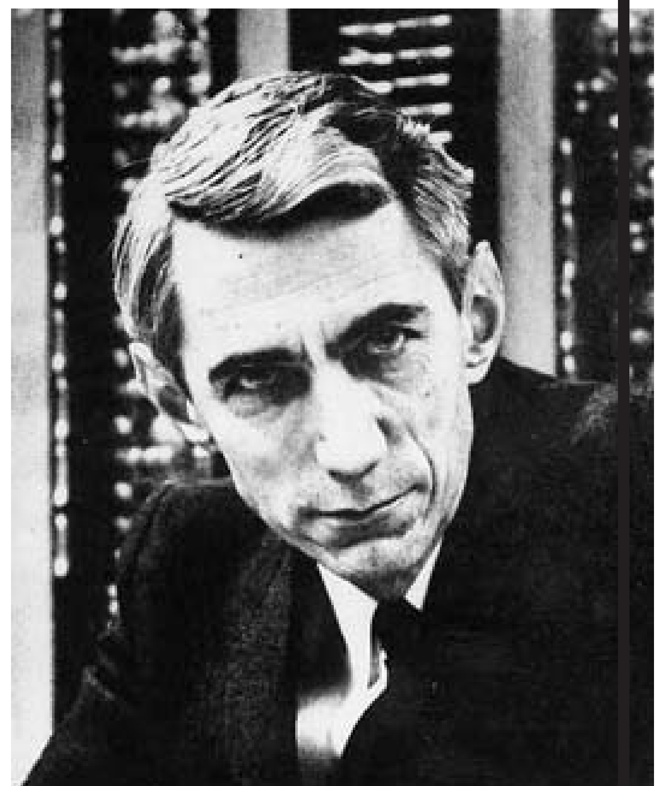
for $n \geq 2$.



n+5 choices
m choices
m choices

Claude Elwood Shannon (1916–2001)

Howard Gardner, "[Shannon's master's thesis is] possibly the most important, and also the most famous, master's thesis of the century."



Comments

- The lower bound $2^n/(2n)$ is rather tight because an upper bound is $n2^n$ (p. 203).
- The proof counted the number of circuits.
  - Some circuits may not be valid at all.
  - Different circuits may also compute the same function.
- Both are fine because we only need an upper bound on the number of circuits.
- We do not need to consider the outdoing edges because they have been counted as incoming edges.

Relations between Complexity Classes

It is, I own, not uncommon to be wrong in theory
and right in practice.
— Edmund Burke (1729–1797),
*A Philosophical Enquiry into the Origin of Our
Ideas of the Sublime and Beautiful* (1757)

Proper (Complexity) Functions

- We say that $f : \mathbb{N} \to \mathbb{N}$ is a **proper (complexity) function** if the following hold:
  - f is nondecreasing.
  - There is a k-string TM M_f such that $M_f(x) = \sqcap^{f(|x|)}$ for any x.[a]
  - M_f halts after $O(|x| + f(|x|))$ steps.
  - M_f uses $O(f(|x|))$ space besides its input x.
- M_f's behavior depends only on $|x|$ not x's contents.
- M_f's running time is bounded by $f(n)$.

[a]The textbook calls "$\sqcap$" the quasi-blank symbol. The use of $M_f(x)$ will become clear in Proposition 17 (p. 221).

Examples of Proper Functions

- Most "reasonable" functions are proper: c, $\lceil \log n \rceil$, polynomials of n, 2^n, $\sqrt{n}$, $n!$, etc.
- If f and g are proper, then so are $f+g$, fg, and 2^g.[a]
- Nonproper functions when serving as the time bounds for complexity classes spoil "the theory building."
  - For example, $\text{TIME}(f(n)) = \text{TIME}(2^{f(n)})$ for some recursive function f (the **gap theorem**).[b]
- Only proper functions f will be used in $\text{TIME}(f(n))$, $\text{SPACE}(f(n))$, $\text{NTIME}(f(n))$, and $\text{NSPACE}(f(n))$.

---

[a] For $f(g)$, we need to add $f(n) \geq n$.
[b] Trakhtenbrot (1964); Borodin (1972).

Precise Turing Machines

- A TM M is **precise** if there are functions f and g such that for every $n \in \mathbb{N}$, for every x of length n, and for every computation path of M,
  - M halts after precisely $f(n)$ steps, and
  - All of its strings are of length precisely $g(n)$ at halting.
    * Recall that if M is a TM with input and output, we exclude the first and last strings.
- M can be deterministic or nondeterministic.

Precise TMs Are General

Proposition 17 *Suppose a TM*[a] M *decides* L *within time (space)* $f(n)$, *where* f *is proper. Then there is a precise TM* M' *which decides* L *in time* $O(n + f(n))$ *(space* $O(f(n))$*, respectively).*

- M' on input x first simulates the TM M_f associated with the proper function f on x.
- M_f's output of length $f(|\,x\,|)$ will serve as a "yardstick" or an "alarm clock."

[a] It can be deterministic or nondeterministic.

The Proof (continued)

- Then M' simulates $M(x)$.
- $M'(x)$ halts when and only when the alarm clock runs out—even if M halts earlier.
- If f is a time bound:
  - The simulation of each step of M on x is matched by advancing the cursor on the "clock" string.
  - Because M' stops at the moment the "clock" string is exhausted—even if $M(x)$ stops earlier, it is precise.
  - The time bound is therefore $O(|\,x\,| + f(|\,x\,|))$.

The Proof (concluded)

- If f is a space bound (sketchy):
  - M' simulates M *on* the quasi-blanks of M_f's output string.
  - The total space, not counting the input string, is $O(f(n))$.
  - But we still need a way to make sure there is no infinite loop.[a]

[a]See the proof of Theorem 24 on p. 239.

Important Complexity Classes

- We write expressions like n^k to denote the union of all complexity classes, one for each value of k.
- For example,

$$\text{NTIME}(n^k) = \bigcup_{j>0} \text{NTIME}(n^j).$$

Important Complexity Classes (concluded)

$$
\begin{aligned}
\mathrm{P} &= \mathrm{TIME}(n^k), \\
\mathrm{NP} &= \mathrm{NTIME}(n^k), \\
\mathrm{PSPACE} &= \mathrm{SPACE}(n^k), \\
\mathrm{NPSPACE} &= \mathrm{NSPACE}(n^k), \\
\mathrm{E} &= \mathrm{TIME}(2^{kn}), \\
\mathrm{EXP} &= \mathrm{TIME}(2^{n^k}), \\
\mathrm{L} &= \mathrm{SPACE}(\log n), \\
\mathrm{NL} &= \mathrm{NSPACE}(\log n).
\end{aligned}
$$

Complements of Nondeterministic Classes

- Recall that the **complement** of L, denoted by $\bar{L}$, is the language $\Sigma^* - L$.
  - SAT COMPLEMENT is the set of unsatisfiable boolean expressions.
- We knew that R, RE, and coRE are distinct (p. 172).
  - Again, coRE contains the complements of *languages* in RE, *not* the languages not in RE.
- How about co$\mathcal{C}$ when $\mathcal{C}$ is a complexity class?

The Co-Classes

- For any complexity class $\mathcal{C}$, co$\mathcal{C}$ denotes the class

$$\{L : \bar{L} \in \mathcal{C}\}.$$

- Clearly, if $\mathcal{C}$ is a *deterministic* time or space *complexity class*, then $\mathcal{C} = \text{co}\mathcal{C}$.
  - They are said to be **closed under complement**.
  - A deterministic TM deciding L can be converted to one that decides $\bar{L}$ within the same time or space bound by reversing the "yes" and "no" states (p. 169).
- Whether nondeterministic classes for time are closed under complement is not known (p. 111).

Comments

- As
$$\text{co}\mathcal{C} = \{L : \bar{L} \in \mathcal{C}\},$$
$L \in \mathcal{C}$ if and only if $\bar{L} \in \text{co}\mathcal{C}$.
- But it is *not* true that $L \in \mathcal{C}$ if and only if $L \notin \text{co}\mathcal{C}$.
  - co$\mathcal{C}$ is not defined as $\bar{\mathcal{C}}$.
- For example, suppose $\mathcal{C} = \{\{2, 4, 6, 8, 10, \ldots\}\}$.
- Then co$\mathcal{C} = \{\{1, 3, 5, 7, 9, \ldots\}\}$.
- But $\bar{\mathcal{C}} = 2^{\{1,2,3,\ldots\}^*} - \{\{2, 4, 6, 8, 10, \ldots\}\}$.

The Quantified Halting Problem

- Let $f(n) \geq n$ be proper.
- Define

$$H_f = \{M; x : M \text{ accepts input } x \text{ after at most } f(|\, x \,|) \text{ steps}\},$$

  where M is deterministic.
- Assume the input is binary.

$H_f \in \mathsf{TIME}(f(n)^3)$

- For each input $M; x$, we simulate M on x with an alarm clock of length $f(|\,x\,|)$.
  - Use the single-string simulator (p. 87), the universal TM (p. 153), and the linear speedup theorem (p. 96).
  - Our simulator accepts $M; x$ if and only if M accepts x before the alarm clock runs out.
- From p. 94, the total running time is $O(\ell_M k_M^2 f(n)^2)$, where ℓ_M is the length to encode each symbol or state of M and k_M is M's number of strings.
- As $\ell_M k_M^2 = O(n)$, the running time is $O(f(n)^3)$, where the constant is independent of M.

$H_f \notin \mathsf{TIME}(f(\lfloor n/2 \rfloor))$

- Suppose TM M_{H_f} decides H_f in time $f(\lfloor n/2 \rfloor)$.
- Consider machine:

$$
\begin{array}{ll}
D_f(M) & \{ \\
& \qquad \textbf{if } M_{H_f}(M;M) = \text{“yes”} \\
& \qquad \textbf{then } \text{“no”}; \\
& \qquad \textbf{else } \text{“yes”}; \\
\} &
\end{array}
$$

- D_f on input M runs in the same time as M_{H_f} on input $M;M$, i.e., in time $f(\lfloor \frac{2n+1}{2} \rfloor) = f(n)$, where $n = |\, M \,|$.[a]

[a]A student pointed out on October 6, 2004, that this estimation omits the time to write down $M;M$.

The Proof (concluded)

- First,

$$\begin{aligned} & D_f(D_f) = \text{“yes”} \\ \Rightarrow \quad & D_f; D_f \notin H_f \\ \Rightarrow \quad & D_f \text{ does not accept } D_f \text{ within time } f(\mid D_f \mid) \\ \Rightarrow \quad & D_f(D_f) \neq \text{“yes”} \\ \Rightarrow \quad & D_f(D_f) = \text{“no”} \end{aligned}$$

  a contradiction

- Similarly, $D_f(D_f) =$ “no” $\Rightarrow D_f(D_f) =$ “yes.”

The Time Hierarchy Theorem

Theorem 18 *If $f(n) \geq n$ is proper, then*

$$\mathrm{TIME}(f(n)) \subsetneq \mathrm{TIME}(f(2n+1)^3).$$

- The quantified halting problem makes it so.

Corollary 19 $\mathrm{P} \subsetneq \mathrm{E}$.

- $\mathrm{P} \subseteq \mathrm{TIME}(2^n)$ because $\mathrm{poly}(n) \leq 2^n$ for n large enough.
- But by Theorem 18,

$$\mathrm{TIME}(2^n) \subsetneq \mathrm{TIME}((2^{2n+1})^3) \subseteq \mathrm{E}.$$

- So $\mathrm{P} \subsetneq \mathrm{E}$.

The Space Hierarchy Theorem

Theorem 20 (Hennie and Stearns (1966)) *If $f(n)$ is proper, then*

$$\mathrm{SPACE}(f(n)) \subsetneq \mathrm{SPACE}(f(n)\log f(n)).$$

Corollary 21 $\mathrm{L} \subsetneq \mathrm{PSPACE}$.

Nondeterministic Time Hierarchy Theorems

Theorem 22 (Cook (1973)) $\text{NTIME}(n^r) \subsetneq \text{NTIME}(n^s)$ *whenever* $1 \leq r < s$.

Theorem 23 (Seiferas, Fischer, and Meyer (1978)) *If* $T_1(n), T_2(n)$ *are proper, then*

$$\text{NTIME}(T_1(n)) \subsetneq \text{NTIME}(T_2(n))$$

whenever $T_1(n+1) = o(T_2(n))$.

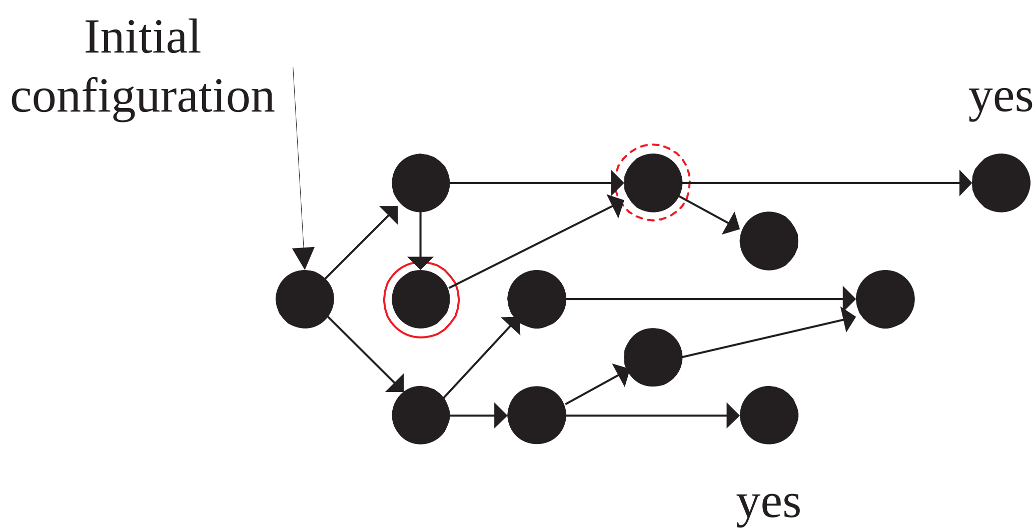
The Reachability Method

- The computation of a time-bounded TM can be represented by a directed graph.
- The TM's configurations constitute the nodes.
- Two nodes are connected by a directed edge if one yields the other in one step.
- The start node representing the initial configuration has zero in degree.

The Reachability Method (concluded)

- When the TM is nondeterministic, a node may have an out degree greater than one.
  - The graph is the same as the computation tree earlier except that identical configuration nodes are merged into one node.
- So M accepts the input if and only if there is a path from the start node to a node with a "yes" state.
- It is the reachability problem.

Illustration of the Reachability Method



Relations between Complexity Classes

Theorem 24 *Suppose $f(n)$ is proper. Then*

1. $\text{SPACE}(f(n)) \subseteq \text{NSPACE}(f(n))$, $\text{TIME}(f(n)) \subseteq \text{NTIME}(f(n))$.
2. $\text{NTIME}(f(n)) \subseteq \text{SPACE}(f(n))$.
3. $\text{NSPACE}(f(n)) \subseteq \text{TIME}(k^{\log n + f(n)})$.

- Proof of 2:
  - Explore the computation *tree* of the NTM for "yes."
  - Specifically, generate an $f(n)$-bit sequence denoting the nondeterministic choices over $f(n)$ steps.

Proof of Theorem 24(2)

- (continued)
  - Simulate the NTM based on the choices.
  - Recycle the space and repeat the above steps.
  - Halt with "yes" when a "yes" is encountered or "no" if the tree is exhausted.
  - Each path simulation consumes at most $O(f(n))$ space because it takes $O(f(n))$ time.
  - The total space is $O(f(n))$ because space is recycled.

Proof of Theorem 24(3)

- Let k-string NTM

$$M = (K, \Sigma, \Delta, s)$$

with input and output decide $L \in \text{NSPACE}(f(n))$.

- Use the reachability method on the configuration graph of M on input x of length n.

- A configuration is a $(2k+1)$-tuple

$$(q, w_1, u_1, w_2, u_2, \ldots, w_k, u_k).$$

Proof of Theorem 24(3) (continued)

- We only care about

$$(q, i, w_2, u_2, \ldots, w_{k-1}, u_{k-1}),$$

where i is an integer between 0 and n for the position of the first cursor.

- The number of configurations is therefore at most

$$|K| \times (n+1) \times |\Sigma|^{(2k-4) f(n)} = O(c_1^{\log n + f(n)}) \tag{2}$$

for some c_1, which depends on M.

- Add edges to the configuration graph based on M's transition function.

Proof of Theorem 24(3) (concluded)

- $x \in L \Leftrightarrow$ there is a path in the configuration graph from the initial configuration to a configuration of the form ("yes", $i, \ldots$).[a]
- This is REACHABILITY on a graph with $O(c_1^{\log n + f(n)})$ nodes.
- It is in TIME$(c^{\log n + f(n)})$ for some c because REACHABILITY $\in$ TIME(n^j) for some j and

$$\left[c_1^{\log n + f(n)} \right]^j = (c_1^j)^{\log n + f(n)}.$$

[a]There may be many of them.

Space-Bounded Computation and Proper Functions

- In the definition of *space-bounded* computations earlier (p. 110), the TMs are not required to halt at all.
- When the space is bounded by a proper function f, computations can be assumed to halt:
  - Run the TM associated with f to produce a quasi-blank output of length $f(n)$ first.
  - The space-bounded computation must repeat a configuration if it runs for more than $c^{\log n + f(n)}$ steps for some c (p. 242).

Space-Bounded Computation and Proper Functions (concluded)

- (continued)
  - So an infinite loop occurs during simulation for a computation path longer than $c^{\log n + f(n)}$ steps.
  - Hence we only simulate up to $c^{\log n + f(n)}$ time steps per computation path.

A Grand Chain of Inclusions[a]

- It is an easy application of Theorem 24 (p. 239) that

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq EXP.$$

- By Corollary 21 (p. 234), we know $L \subsetneq PSPACE$.
- So the chain must break somewhere between L and EXP.
- It is suspected that all four inclusions are proper.
- But there are no proofs yet.

[a]With input from Mr. Chin-Luei Chang (R93922004, D95922007) on October 22, 2004.

Nondeterministic Space and Deterministic Space

- By Theorem 5 (p. 116),

$$\text{NTIME}(f(n)) \subseteq \text{TIME}(c^{f(n)}),$$

  an exponential gap.

- There is no proof yet that the exponential gap is inherent.
- How about NSPACE vs. SPACE?
- Surprisingly, the relation is only quadratic—a polynomial—by Savitch's theorem.