Space Complexity

- Consider a k-string TM M with input x.
- Assume non-$\sqcup$ is never written over by $\sqcup$.[a]
  - The purpose is not to artificially reduce the space needs (see below).
- If M halts in configuration

$$(H, w_1, u_1, w_2, u_2, \ldots, w_k, u_k),$$

then the **space required by M on input x** is

$$\sum_{i=1}^{k} |w_i u_i|.$$

[a]Corrected by Ms. Chuan-Ju Wang (R95922018, F95922018) on September 27, 2006.

Space Complexity (continued)

- Suppose we do not charge the space used only for input and output.
- Let $k > 2$ be an integer.
- A **k-string Turing machine with input and output** is a k-string TM that satisfies the following conditions.
  - The input string is *read-only*.
  - The last string, the output string, is *write-only*.
  - So the cursor never moves to the left.
  - The cursor of the input string does not wander off into the $\sqcup$s.

Space Complexity (concluded)

- If M is a TM with input and output, then the space required by M on input x is

$$\sum_{i=2}^{k-1} |w_i u_i|.$$

- Machine M **operates within space bound** $f(n)$ for $f : \mathbb{N} \to \mathbb{N}$ if for any input x, the space required by M on x is at most $f(|\, x\,|)$.

Space Complexity Classes

- Let L be a language.
- Then
$$L \in \mathrm{SPACE}(f(n))$$
if there is a TM with input and output that decides L and operates within space bound $f(n)$.
- $\mathrm{SPACE}(f(n))$ is a set of languages.
  - PALINDROME $\in \mathrm{SPACE}(\log n)$.[a]
- As in the linear speedup theorem (p. 96), constant coefficients do not matter.

[a] Keep 3 counters.

Nondeterminism[a]

- A **nondeterministic Turing machine** (**NTM**) is a quadruple $N = (K, \Sigma, \Delta, s)$.
- K, Σ, s are as before.
- $\Delta \subseteq K \times \Sigma \times (K \cup \{h, \text{"yes"}, \text{"no"}\}) \times \Sigma \times \{\leftarrow, \rightarrow, -\}$ is a relation, not a function.[b]
  - For each state-symbol combination (q, σ), there may be multiple valid next steps.
  - Multiple lines of code may be applicable.

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[a] Rabin and Scott (1959).

[b] Corrected by Mr. Jung-Ying Chen (`D95723006`) on September 23, 2008.

Nondeterminism (concluded)

- As before, a program contains lines of code:

$$\begin{aligned}(q_1, \sigma_1, p_1, \rho_1, D_1) &\in \Delta,\\(q_2, \sigma_2, p_2, \rho_2, D_2) &\in \Delta,\\&\vdots\\(q_n, \sigma_n, p_n, \rho_n, D_n) &\in \Delta.\end{aligned}$$

  - We cannot write

$$\delta(q_i, \sigma_i) = (p_i, \rho_i, D_i)$$

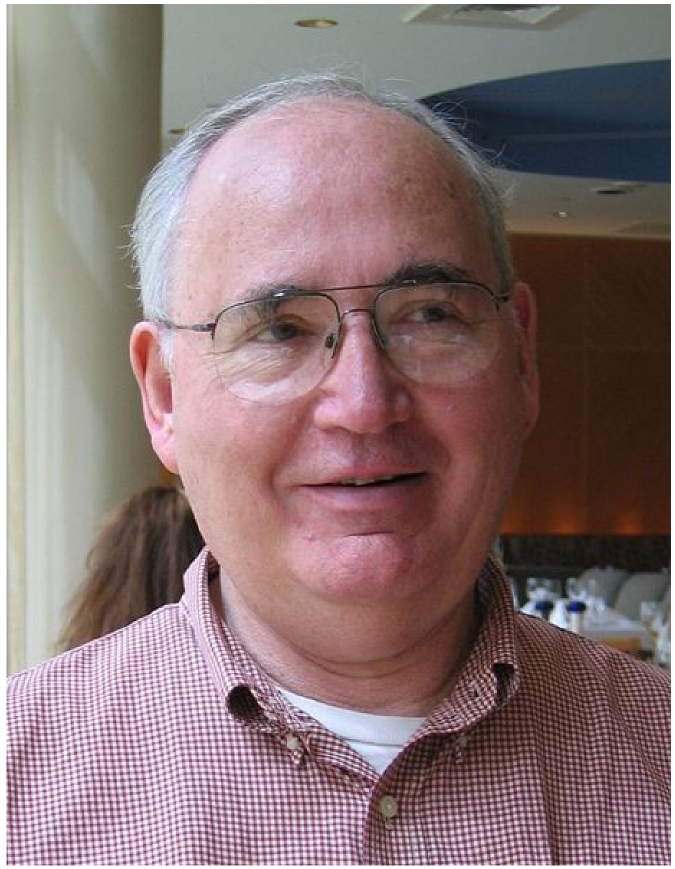
    as in the deterministic case (p. 22) anymore.

- A configuration yields another configuration in one step if there *exists* a rule in Δ that makes this happen.

Michael O. Rabin[a] (1931–)

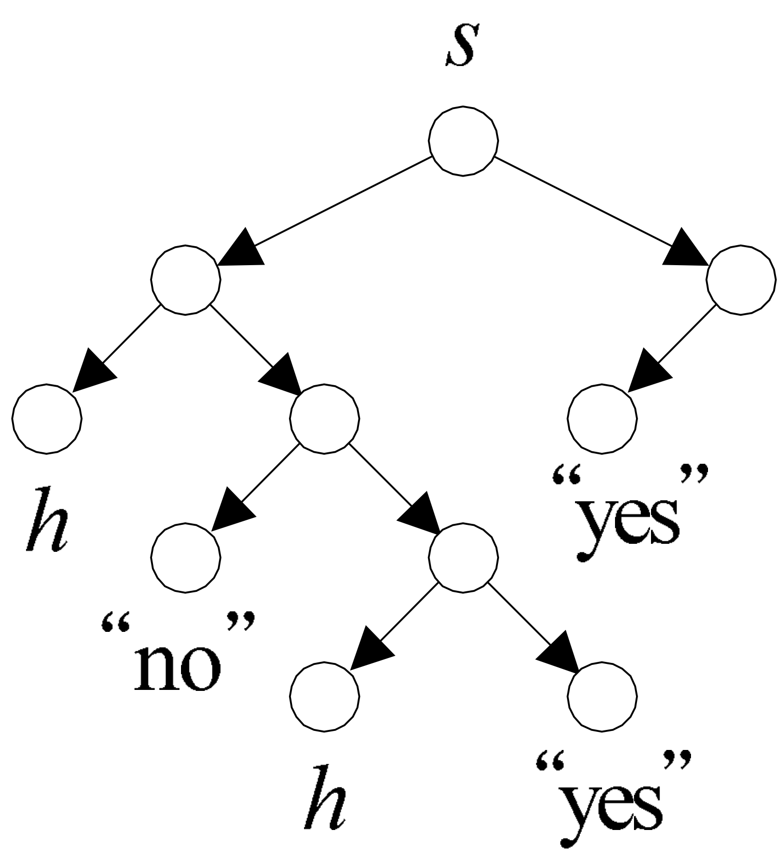
[a]Turing Award (1976).

Dana Stewart Scott[a] (1932–)



[a]Turing Award (1976).

Computation Tree and Computation Path



Decidability under Nondeterminism

- Let L be a language and N be an NTM.
- N **decides** L if for any $x \in \Sigma^*$, $x \in L$ if and only if there is a sequence of valid configurations that ends in "yes."
- In other words,
  - If $x \in L$, then $N(x) =$ "yes" for some computation path.
  - If $x \notin L$, then $N(x) \neq$ "yes" for all computation paths.

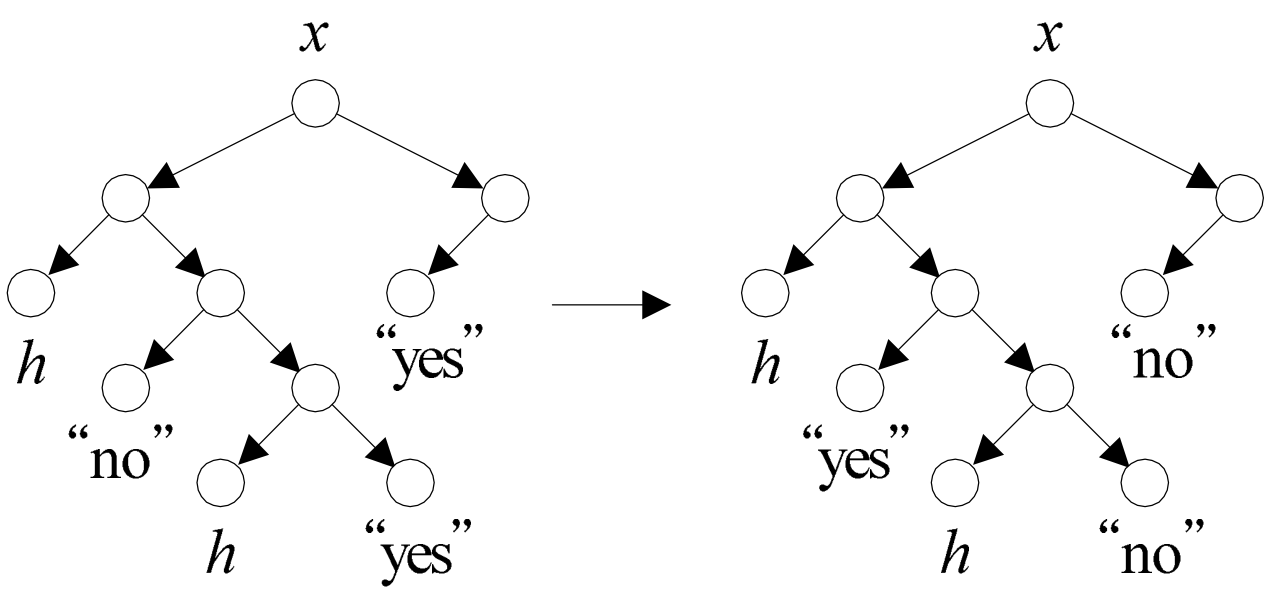
Decidability under Nondeterminism (concluded)

- It is not required that the NTM halts in all computation paths.[a]
- If $x \notin L$, no nondeterministic choices should lead to a "yes" state.
- The key is the algorithm's *overall* behavior not whether it gives a correct answer for each particular run.
- Determinism is a special case of nondeterminism.

[a] So "accepts" is a more proper term, and other books use "decides" only when the NTM always halts.

Complementing a TM's Halting States

- Let M decide L, and M' be M after "yes" $\leftrightarrow$ "no".
- If M is a deterministic TM, then M' decides $\bar{L}$.
  - So M and M' decide languages that are complements of each other.
- But if M is an NTM, then M' may not decide $\bar{L}$.
  - It is possible that both M and M' accept x (see next page).
  - So M and M' accept languages that are not complements of each other.


x
h
"no"
h
"yes"
"yes"
x
h
"yes"
h
"no"
"no"

Time Complexity under Nondeterminism

- Nondeterministic machine N decides L **in time** $f(n)$, where $f : \mathbb{N} \to \mathbb{N}$, if
  - N decides L, and
  - for any $x \in \Sigma^*$, N does not have a computation path longer than $f(|\, x \,|)$.
- We charge only the "depth" of the computation tree.

Time Complexity Classes under Nondeterminism

- NTIME($f(n)$) is the set of languages decided by NTMs within time $f(n)$.
- NTIME($f(n)$) is a complexity class.

NP

- Define
$$\text{NP} = \bigcup_{k>0} \text{NTIME}(n^k).$$
- Clearly $\text{P} \subseteq \text{NP}$.
- Think of NP as efficiently *verifiable* problems (see p. 326).
  - Boolean satisfiability (p. 119 and p. 196).
- The most important open problem in computer science is whether $\text{P} = \text{NP}$.

Simulating Nondeterministic TMs

Nondeterminism does not add power to TMs.

Theorem 5 *Suppose language L is decided by an NTM N in time $f(n)$. Then it is decided by a 3-string deterministic TM M in time $O(c^{f(n)})$, where $c > 1$ is some constant depending on N.*

- On input x, M goes down every computation path of N using depth-first search.
  - M does *not* need to know $f(n)$.
  - As N is time-bounded, the depth-first search will not run indefinitely.

The Proof (concluded)

- If any path leads to "yes," then M immediately enters the "yes" state.
- If none of the paths leads to "yes," then M enters the "no" state.
- The simulation takes time $O(c^{f(n)})$ for some $c > 1$ because the computation tree has that many nodes.

Corollary 6 $\text{NTIME}(f(n))) \subseteq \bigcup_{c>1} \text{TIME}(c^{f(n)})$.

NTIME vs. TIME

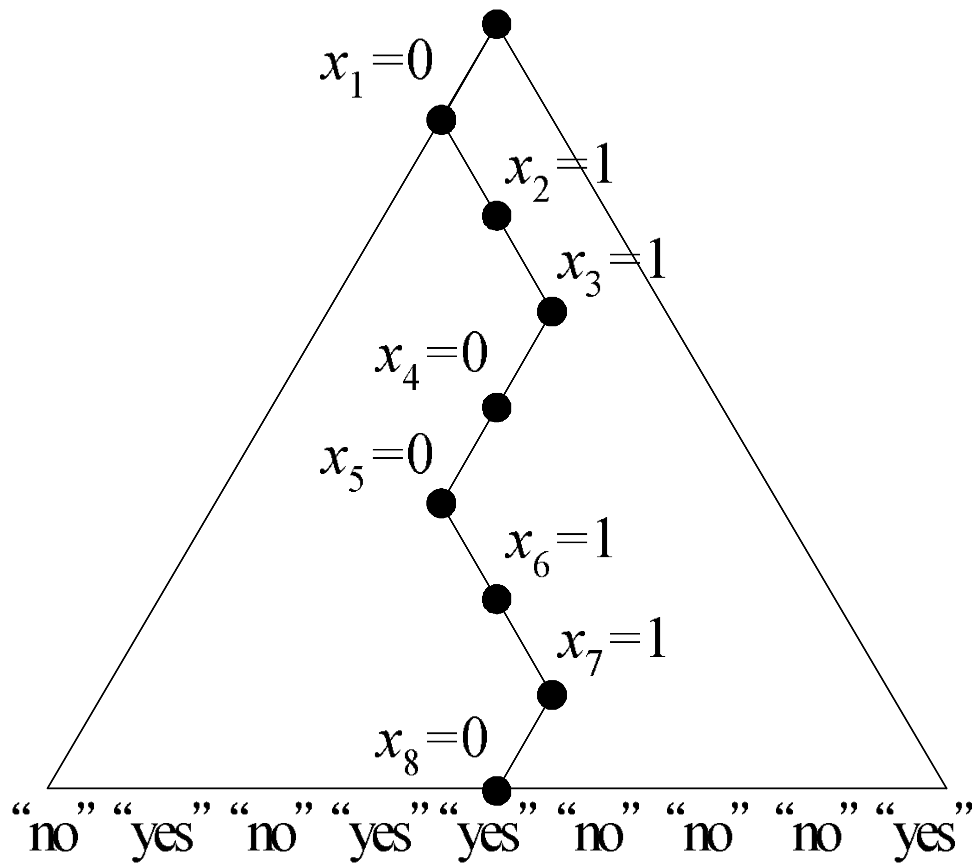
- Does converting an NTM into a TM require exploring all computation paths of the NTM as done in Theorem 5 (p. 116)?
- This is the most important question in theory with important practical implications.

A Nondeterministic Algorithm for Satisfiability

ϕ is a boolean formula with n variables.

for $i = 1, 2, \ldots, n$ **do**
    Guess $x_i \in \{0, 1\}$; {Nondeterministic choice.}
end for
{Verification:}
if $\phi(x_1, x_2, \ldots, x_n) = 1$ **then**
    "yes";
else
    "no";
end if

Computation Tree for Satisfiability



Analysis

- The computation tree is a complete binary tree of depth n.
- Every computation path corresponds to a particular truth assignment out of 2^n.
- ϕ is satisfiable iff there is a truth assignment that satisfies ϕ.

Analysis (concluded)

- The algorithm decides language $\{\phi : \phi \text{ is satisfiable}\}$.
  - Suppose ϕ is satisfiable.
    - That means there is a truth assignment that satisfies ϕ.
    - So there is a computation path that results in "yes."
  - Suppose ϕ is not satisfiable.
    - That means every truth assignment makes ϕ false.
    - So every computation path results in "no."
- General paradigm: Guess a "proof" and then verify it.

The Traveling Salesman Problem

- We are given n cities $1, 2, \ldots, n$ and integer distance d_{ij} between any two cities i and j.
- Assume $d_{ij} = d_{ji}$ for convenience.
- The **traveling salesman problem** (TSP) asks for the total distance of the shortest tour of the cities.
- The decision version TSP (D) asks if there is a tour with a total distance at most B, where B is an input.[a]

---

[a]Both problems are extremely important and are equally hard (p. 389 and p. 490).

A Nondeterministic Algorithm for TSP (D)

for $i = 1, 2, \ldots, n$ **do**
    Guess $x_i \in \{1, 2, \ldots, n\}$; {The ith city.}[a]
end for
$x_{n+1} := x_1$;
{Verification:}
if $x_1, x_2, \ldots, x_n$ are distinct and $\sum_{i=1}^{n} d_{x_i, x_{i+1}} \leq B$ **then**
    "yes";
else
    "no";
end if

---

[a] Can be made into a series of $\log_2 n$ *binary* choices for each x_i so that the next-state count (2) is a constant, independent of input size. Contributed by Mr. Chih-Duo Hong (R95922079) on September 27, 2006.

Analysis

- Suppose the input graph contains at least one tour of the cities with a total distance at most B.
  - Then there is a computation path that leads to "yes."[a]
- Suppose the input graph contains no tour of the cities with a total distance at most B.
  - Then every computation path leads to "no."

[a]It does not mean the algorithm will follow that path. It just means such a computation path exists.

Remarks on the $\mathsf{P} \stackrel{?}{=} \mathsf{NP}$ Open Problem[a]

- Many practical applications depend on answers to the $P \stackrel{?}{=} NP$ question.
- Verification of password is easy (so it is in NP).
  - A computer should not take a long time to let a user log in.
- A password system should be hard to crack (loosely speaking, cracking it should not be in P).
- It took mathematicians and logicians 63 years to settle the Continuum Hypothesis.

[a] Contributed by Mr. Kuan-Lin Huang (B96902079, R00922018) on September 27, 2011.

Nondeterministic Space Complexity Classes

- Let L be a language.
- Then
$$L \in \text{NSPACE}(f(n))$$
if there is an NTM with input and output that decides L and operates within space bound $f(n)$.
- $\text{NSPACE}(f(n))$ is a set of languages.
- As in the linear speedup theorem (Theorem 4 on p. 96), constant coefficients do not matter.

Graph Reachability

- Let $G(V, E)$ be a directed graph (**digraph**).
- REACHABILITY asks if, given nodes a and b, does G contain a path from a to b?
- Can be easily solved in polynomial time by breadth-first search.
- How about its nondeterministic space complexity?

The First Try: NSPACE($n \log n$)

Determine the number of nodes m; {Note $m \leq n$.}
$x_1 := a$; {Assume $a \neq b$.}
for $i = 2, 3, \ldots, m$ **do**
    Guess $x_i \in \{v_1, v_2, \ldots, v_m\}$; {The ith node.}
end for
for $i = 2, 3, \ldots, m$ **do**
    if $(x_{i-1}, x_i) \notin E$ **then**
        "no";
    end if
    if $x_i = b$ **then**
        "yes";
    end if
end for
"no";

In Fact, REACHABILITY $\in$ NSPACE$(\log n)$

Determine the number of nodes m; $\{$Note $m \leq n$.$\}$

$x := a$;

for $i = 2, 3, \ldots, m$ **do**

Guess $y \in \{v_1, v_2, \ldots, v_m\}$; $\{$The next node.$\}$

if $(x, y) \notin E$ **then**

"no";

end if

if $y = b$ **then**

"yes";

end if

$x := y$;

end for

"no";

Space Analysis

- Variables m, i, x, and y each require $O(\log n)$ bits.
- Testing $(x, y) \in E$ is accomplished by consulting the input string with counters of $O(\log n)$ bits long.
- Hence

$$\text{REACHABILITY} \in \text{NSPACE}(\log n).$$

  - REACHABILITY with more than one terminal node also has the same complexity.
- REACHABILITY $\in$ P (p. 239).

Undecidability

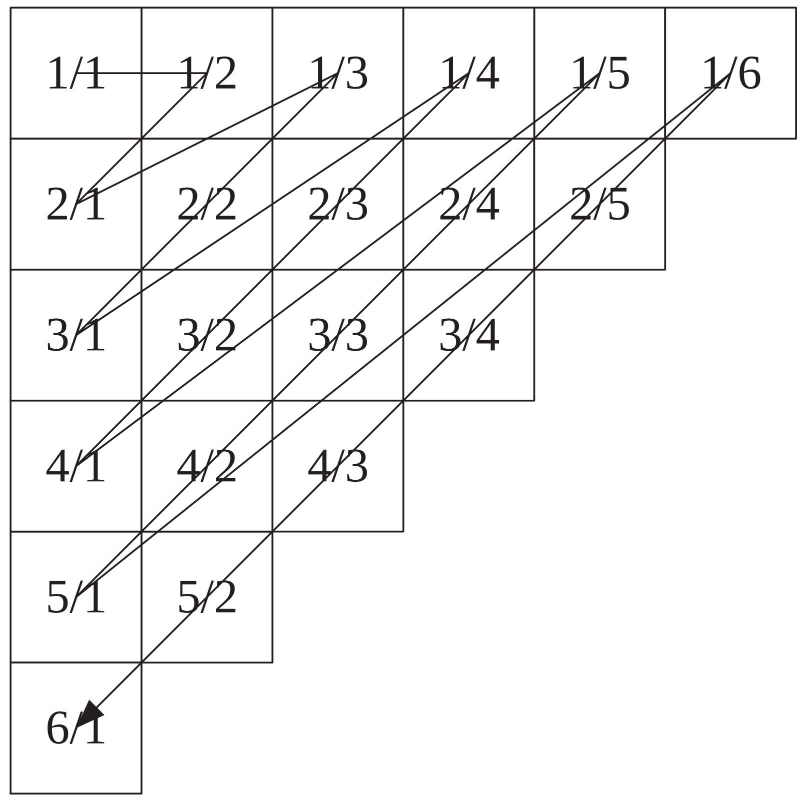
God exists since mathematics is consistent,
and the Devil exists since we cannot prove it.
— André Weil (1906–1998)

Whatsoever we imagine is *finite*.
Therefore there is no idea, or conception
of any thing we call *infinite*.
— Thomas Hobbes (1588–1679), *Leviathan*

Infinite Sets

- A set is **countable** if it is finite or if it can be put in one-one correspondence with $\mathbb{N} = \{ 0, 1, \ldots \}$, the set of natural numbers.
  - Set of integers $\mathbb{Z}$.
    - $0 \leftrightarrow 0$.
    - $1 \leftrightarrow 1, 2 \leftrightarrow 3, 3 \leftrightarrow 5, \ldots$.
    - $-1 \leftrightarrow 2, -2 \leftrightarrow 4, -3 \leftrightarrow 6, \ldots$.
  - Set of positive integers $\mathbb{Z}^+$: $i \leftrightarrow i - 1$.
  - Set of positive odd integers: $i \leftrightarrow (i - 1)/2$.
  - Set of (positive) rational numbers: See next page.
  - Set of squared integers: $i \leftrightarrow \sqrt{i}$.

Rational Numbers Are Countable



Cardinality

- For any set A, define $|A|$ as A's **cardinality** (size).
- Two sets are said to have the same cardinality, or

$$|A| = |B| \quad \text{or} \quad A \sim B,$$

  if there exists a one-to-one correspondence between their elements.
- 2^A denotes set A's **power set**, that is $\{B : B \subseteq A\}$.
  - E.g., $\{0, 1\}$'s power set is $2^{\{0,1\}} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$.
- If $|A| = k$, then $|2^A| = 2^k$.

Cardinality (concluded)

- Define $|A| \leq |B|$ if there is a one-to-one correspondence between A and a subset of B's.
- Obviously, if $A \subseteq B$, then $|A| \leq |B|$.
  - So $|\,\mathbb{N}\,| \leq |\,\mathbb{Z}\,|$.
  - So $|\,\mathbb{N}\,| \leq |\,\mathbb{R}\,|$.
- Define $|A| < |B|$ if $|A| \leq |B|$ but $|A| \neq |B|$.
- If $A \subsetneq B$, then $|A| < |B|$?

Cardinality and Infinite Sets

- If A and B are infinite sets, it is possible that $A \subsetneq B$ yet $|A| = |B|$.
  - The set of integers *properly* contains the set of odd integers.
  - But the set of integers has the same cardinality as the set of odd integers (p. 134).[a]
- A lot of "paradoxes."

[a] Leibniz uses it to "prove" that there are no infinite numbers (Russell, 1914).

Galileo's[a] Paradox (1638)

- The squares of the positive integers can be placed in one-to-one correspondence with all the positive integers.
- This is contrary to the axiom of Euclid[b] that the whole is greater than any of its proper parts.[c]
- Resolution of paradoxes: Pick the notion that results in "better" mathematics.
- The difference between a mathematical paradox and a contradiction is often a matter of opinions.

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[a] Galileo (1564–1642).
[b] Euclid (325 B.C.–265 B.C.).
[c] Leibniz never challenges that axiom (Knobloch, 1999).

Hilbert's[a] Paradox of the Grand Hotel

- For a hotel with a finite number of rooms with all the rooms occupied, a new guest will be turned away.
- Now imagine a hotel with an infinite number of rooms, all of which are occupied.
- A new guest comes and asks for a room.
- "But of course!" exclaims the proprietor.
- He moves the person previously occupying Room 1 to Room 2, the person from Room 2 to Room 3, and so on.
- The new customer now occupies Room 1.

[a] David Hilbert (1862–1943).

Hilbert's Paradox of the Grand Hotel (concluded)

- Now imagine a hotel with an infinite number of rooms, all taken up.
- An infinite number of new guests come in and ask for rooms.
- "Certainly," says the proprietor.
- He moves the occupant of Room 1 to Room 2, the occupant of Room 2 to Room 4, and so on.
- Now all odd-numbered rooms become free and the infinity of new guests can be accommodated in them.
- "There are many rooms in my Father's house, and I am going to prepare a place for you." (*John* 14:3)

David Hilbert (1862–1943)

The point of philosophy is
to start with something so simple
as not to seem worth stating,
and to end with something
so paradoxical that no one will believe it.
— Bertrand Russell (1872–1970)