Boolean Logic

It seemed unworthy of a grown man
to spend his time on such trivialities,
but what was I to do? $[\cdots]$
The whole of the rest of my life might be
consumed in looking at
that blank sheet of paper.
— Bertrand Russell (1872–1970),
Autobiography, Vol. I (1967)

Boolean Logic[a]

Boolean variables: $x_1, x_2, \ldots$.

Literals: x_i, $\neg x_i$.

Boolean connectives: $\lor, \land, \neg$.

Boolean expressions: Boolean variables, $\neg\phi$ (**negation**), $\phi_1 \lor \phi_2$ (**disjunction**), $\phi_1 \land \phi_2$ (**conjunction**).

- $\bigvee_{i=1}^{n} \phi_i$ stands for $\phi_1 \lor \phi_2 \lor \cdots \lor \phi_n$.
- $\bigwedge_{i=1}^{n} \phi_i$ stands for $\phi_1 \land \phi_2 \land \cdots \land \phi_n$.

Implications: $\phi_1 \Rightarrow \phi_2$ is a shorthand for $\neg\phi_1 \lor \phi_2$.

Biconditionals: $\phi_1 \Leftrightarrow \phi_2$ is a shorthand for $(\phi_1 \Rightarrow \phi_2) \land (\phi_2 \Rightarrow \phi_1)$.

[a]George Boole (1815–1864) in 1847.

Truth Assignments

- A **truth assignment** T is a mapping from boolean variables to **truth values** `true` and `false`.
- A truth assignment is **appropriate** to boolean expression ϕ if it defines the truth value for every variable in ϕ.
  - $\{x_1 = \texttt{true}, x_2 = \texttt{false}\}$ is appropriate to $x_1 \lor x_2$.
  - $\{x_2 = \texttt{true}, x_3 = \texttt{false}\}$ is not appropriate to $x_1 \lor x_2$.

Satisfaction

- $T \models \phi$ means boolean expression ϕ is true under T; in other words, T **satisfies** ϕ.
- ϕ_1 and ϕ_2 are **equivalent**, written
$$\phi_1 \equiv \phi_2,$$
if for any truth assignment T appropriate to both of them, $T \models \phi_1$ if and only if $T \models \phi_2$.

Truth Tables

- Suppose ϕ has n boolean variables.
- A **truth table** contains 2^n rows.
- Each row corresponds to one truth assignment of the n variables and records the truth value of ϕ under that truth assignment.
- A truth table can be used to prove if two boolean expressions are equivalent.
  - Just check if they give identical truth values under all appropriate truth assignments.

A Truth Table

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

A Second Truth Table

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

A Third Truth Table

p	$\neg p$
0	1
1	0

De Morgan's Laws[a]

- **De Morgan's laws** say that

$$\begin{aligned} \neg(\phi_1 \land \phi_2) &\equiv \neg\phi_1 \lor \neg\phi_2, \\ \neg(\phi_1 \lor \phi_2) &\equiv \neg\phi_1 \land \neg\phi_2. \end{aligned}$$

- Here is a proof of the first law:

ϕ_1	ϕ_2	$\neg(\phi_1 \land \phi_2)$	$\neg\phi_1 \lor \neg\phi_2$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

[a] Augustus DeMorgan (1806–1871) or William of Ockham (1288–1348).

Conjunctive Normal Forms

- A boolean expression ϕ is in **conjunctive normal form** (**CNF**) if
$$\phi = \bigwedge_{i=1}^{n} C_i,$$
where each **clause** C_i is the disjunction of zero or more literals.[a]
  - For example,
$$(x_1 \lor x_2) \land (x_1 \lor \neg x_2) \land (x_2 \lor x_3).$$
- Convention: An empty CNF is satisfiable, but a CNF containing an empty clause is not.

[a] Improved by Mr. Aufbu Huang (`R95922070`) on October 5, 2006.

Disjunctive Normal Forms

- A boolean expression ϕ is in **disjunctive normal form** (**DNF**) if

$$\phi = \bigvee_{i=1}^{n} D_i,$$

where each **implicant** D_i is the conjunction of zero or more literals.

  – For example,

$$(x_1 \land x_2) \lor (x_1 \land \neg x_2) \lor (x_2 \land x_3).$$

Clauses and Implicants

- The $\bigvee$ of clauses remains a clause.
  - For example,

$$\begin{aligned} & (x_1 \lor x_2) \lor (x_1 \lor \neg x_2) \lor (x_2 \lor x_3) \\ = \quad & x_1 \lor x_2 \lor x_1 \lor \neg x_2 \lor x_2 \lor x_3. \end{aligned}$$

- The $\bigwedge$ of implicants remains a implicant.
  - For example,

$$\begin{aligned} & (x_1 \land x_2) \land (x_1 \land \neg x_2) \land (x_2 \land x_3) \\ = \quad & x_1 \land x_2 \land x_1 \land \neg x_2 \land x_2 \land x_3. \end{aligned}$$

Any Expression ϕ Can Be Converted into CNFs and DNFs

$\phi = x_j$**:**

- This is trivially true.

$\phi = \neg\phi_1$ **and a CNF is sought:**

- Turn ϕ_1 into a DNF.
- Apply de Morgan's laws to make a CNF for ϕ.

$\phi = \neg\phi_1$ **and a DNF is sought:**

- Turn ϕ_1 into a CNF.
- Apply de Morgan's laws to make a DNF for ϕ.

Any Expression ϕ Can Be Converted into CNFs and DNFs (continued)

$\phi = \phi_1 \vee \phi_2$ and a DNF is sought:

- Make ϕ_1 and ϕ_2 DNFs.

$\phi = \phi_1 \vee \phi_2$ and a CNF is sought:

- Turn ϕ_1 and ϕ_2 into CNFs,[a]

$$\phi_1 = \bigwedge_{i=1}^{n_1} A_i, \quad \phi_2 = \bigwedge_{j=1}^{n_2} B_j.$$

- Set

$$\phi = \bigwedge_{i=1}^{n_1} \bigwedge_{j=1}^{n_2} (A_i \vee B_j).$$

[a]Corrected by Mr. Chun-Jie Yang (`R99922150`) on November 9, 2010.

Any Expression ϕ Can Be Converted into CNFs and DNFs (concluded)

$\phi = \phi_1 \wedge \phi_2$ **and a CNF is sought:**

- Make ϕ_1 and ϕ_2 CNFs.

$\phi = \phi_1 \wedge \phi_2$ **and a DNF is sought:**

- Turn ϕ_1 and ϕ_2 into DNFs,

$$\phi_1 = \bigvee_{i=1}^{n_1} A_i, \quad \phi_2 = \bigvee_{j=1}^{n_2} B_j.$$

- Set

$$\phi = \bigvee_{i=1}^{n_1} \bigvee_{j=1}^{n_2} (A_i \wedge B_j).$$

An Example: Turn $\neg((a \land y) \lor (z \lor w))$ into a DNF

$$
\begin{aligned}
& \neg((a \land y) \lor (z \lor w)) \\
\overset{\neg(\text{CNF}\lor\text{CNF})}{=} \quad & \neg(((a) \land (y)) \lor ((z \lor w))) \\
\overset{\neg(\text{CNF})}{=} \quad & \neg((a \lor z \lor w) \land (y \lor z \lor w)) \\
\overset{\text{de Morgan}}{=} \quad & \neg(a \lor z \lor w) \lor \neg(y \lor z \lor w) \\
\overset{\text{de Morgan}}{=} \quad & (\neg a \land \neg z \land \neg w) \lor (\neg y \land \neg z \land \neg w).
\end{aligned}
$$

Satisfiability

- A boolean expression ϕ is **satisfiable** if there is a truth assignment T appropriate to it such that $T \models \phi$.
- ϕ is **valid** or a **tautology**,[a] written $\models \phi$, if $T \models \phi$ for all T appropriate to ϕ.
- ϕ is **unsatisfiable** if and only if ϕ is false under all appropriate truth assignments if and only if $\neg\phi$ is valid.

[a] Wittgenstein (1889–1951) in 1922. Wittgenstein is one of the most important philosophers of all time. "God has arrived," the great economist Keynes (1883–1946) said of him on January 18, 1928. "I met him on the 5:15 train." Russell (1919), "The importance of 'tautology' for a definition of mathematics was pointed out to me by my former pupil Ludwig Wittgenstein, who was working on the problem. I do not know whether he has solved it, or even whether he is alive or dead."

Ludwig Wittgenstein (1889–1951)

Wittgenstein (1922), “Whereof one cannot speak, thereof one must be silent.”

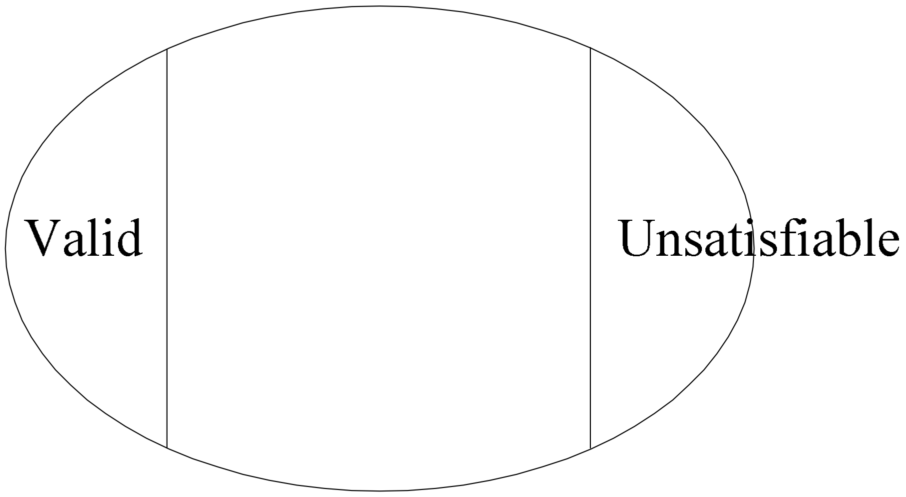
SATISFIABILITY (SAT)

- The **length** of a boolean expression is the length of the string encoding it.
- SATISFIABILITY (SAT): Given a CNF ϕ, is it satisfiable?
- Solvable in exponential time on a TM by the truth table method.
- Solvable in polynomial time on an NTM, hence in NP (p. 104).
- A most important problem in settling the "P $\overset{?}{=}$ NP" problem (p. 294).

UNSATISFIABILITY (UNSAT or SAT COMPLEMENT) and VALIDITY

- UNSAT (SAT COMPLEMENT): Given a boolean expression ϕ, is it unsatisfiable?
- VALIDITY: Given a boolean expression ϕ, is it valid?
  - ϕ is valid if and only if $\neg\phi$ is unsatisfiable.
  - ϕ and $\neg\phi$ are basically of the same length.
  - So UNSAT and VALIDITY have the same complexity.
- Both are solvable in exponential time on a TM by the truth table method.
- Can we do better?

Relations among SAT, UNSAT, and VALIDITY



- The negation of an unsatisfiable expression is a valid expression.
- None of the three problems—satisfiability, unsatisfiability, validity—are known to be in P.

Boolean Functions

- An n-ary boolean function is a function

$$f : \{\texttt{true}, \texttt{false}\}^n \to \{\texttt{true}, \texttt{false}\}.$$

- It can be represented by a truth table.
- There are 2^{2^n} such boolean functions.
  - We can assign `true` or `false` to f for each of the 2^n truth assignments.

Boolean Functions (continued)

Assignment	Truth value
1	`true` or `false`
2	`true` or `false`
$\vdots$	$\vdots$
2^n	`true` or `false`

Boolean Functions (continued)

- A boolean expression expresses a boolean function.
  - Think of its truth value under all truth assignments.
- A boolean function expresses a boolean expression.
  - $\bigvee_{T \models \phi,\text{ literal } y_i \text{ is true in ``row'' } T} (y_1 \wedge \cdots \wedge y_n)$.
    - $y_1 \wedge \cdots \wedge y_n$ is called the **minterm** over $\{x_1, \ldots, x_n\}$ for T.[a]
  - The size[b] is $\leq n2^n \leq 2^{2n}$.

[a] Similar to **programmable logic array**.

[b] We count only the literals here.

Boolean Functions (continued)

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

The corresponding boolean expression:

$$(\neg x_1 \land \neg x_2) \lor (\neg x_1 \land x_2) \lor (x_1 \land x_2).$$

Boolean Functions (concluded)

Corollary 14 *Every n-ary boolean function can be expressed by a boolean expression of size $O(n2^n)$.*

- In general, the exponential length in n cannot be avoided (p. 194).
- The size of the truth table is also $O(n2^n)$.

Boolean Circuits

- A **boolean circuit** is a graph C whose nodes are the **gates**.
- There are no cycles in C.
- All nodes have indegree (number of incoming edges) equal to 0, 1, or 2.
- Each gate has a **sort** from

$$\{\texttt{true}, \texttt{false}, \vee, \wedge, \neg, x_1, x_2, \ldots\}.$$

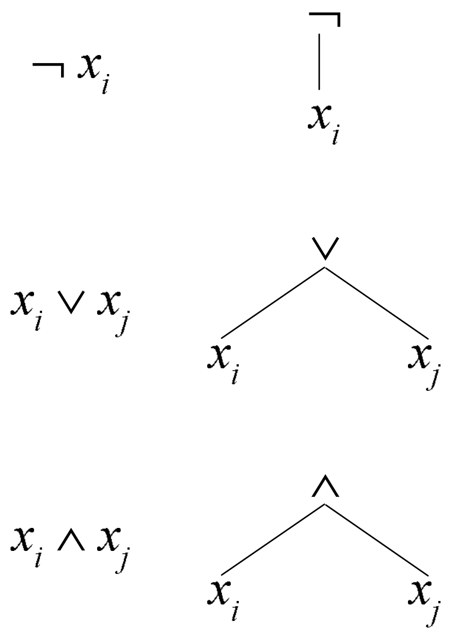
  – There are $n + 5$ sorts.

Boolean Circuits (concluded)

- Gates with a sort from $\{\texttt{true}, \texttt{false}, x_1, x_2, \ldots\}$ are the **inputs** of C and have an indegree of zero.
- The **output gate**(s) has no outgoing edges.
- A boolean circuit computes a boolean function.
- The same boolean function can be computed by infinitely many equivalent boolean circuits.

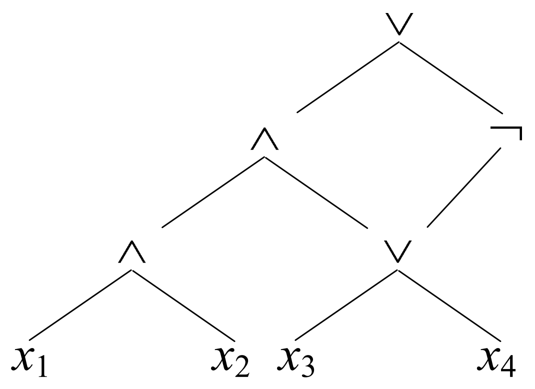
Boolean Circuits and Expressions

- They are equivalent representations.
- One can construct one from the other:



An Example

$$((x_1 \wedge x_2) \wedge (x_3 \vee x_4)) \vee (\neg(x_3 \vee x_4))$$



- Circuits are more economical because of the possibility of sharing.

CIRCUIT SAT and CIRCUIT VALUE

CIRCUIT SAT: Given a circuit, is there a truth assignment such that the circuit outputs true?

- CIRCUIT SAT $\in$ NP: Guess a truth assignment and then evaluate the circuit.

CIRCUIT VALUE: The same as CIRCUIT SAT except that the circuit has no variable gates.

- CIRCUIT VALUE $\in$ P: Evaluate the circuit from the input gates gradually towards the output gate.

Some Boolean Functions Need Exponential Circuits[a]

Theorem 15 (Shannon (1949)) *For any $n \geq 2$, there is an n-ary boolean function f such that no boolean circuits with $2^n/(2n)$ or fewer gates can compute it.*

- There are 2^{2^n} different n-ary boolean functions (p. 184).
- So it suffices to prove that the number of boolean circuits with $2^n/(2n)$ or fewer gates is less than 2^{2^n}.

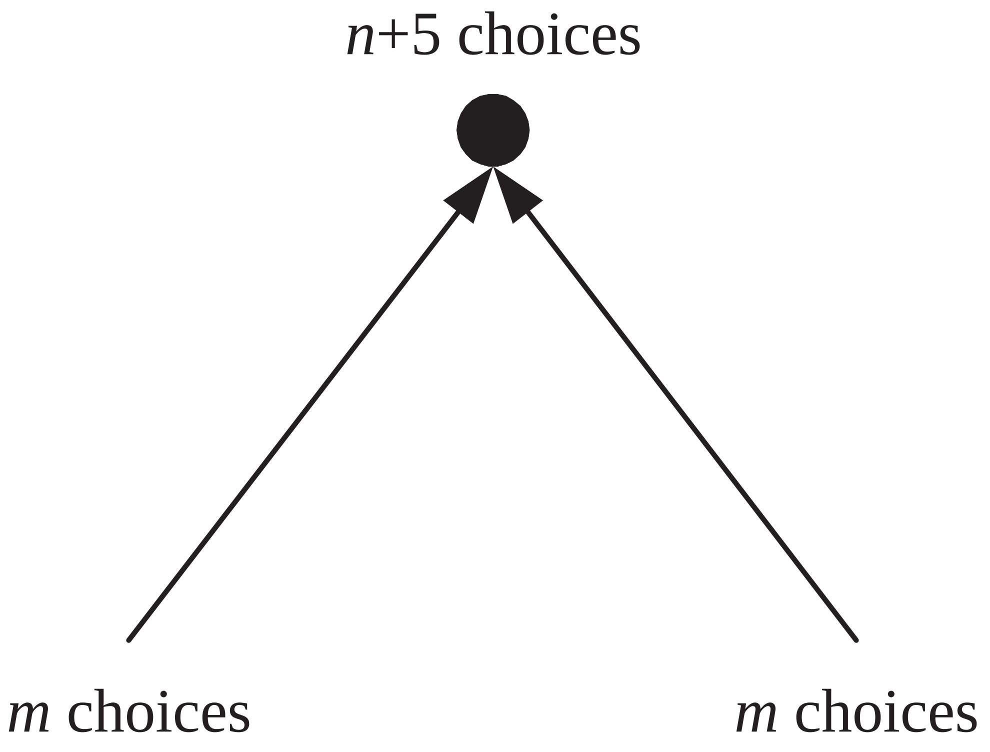
[a]Can be strengthened to "almost all boolean functions . . ."

The Proof (concluded)

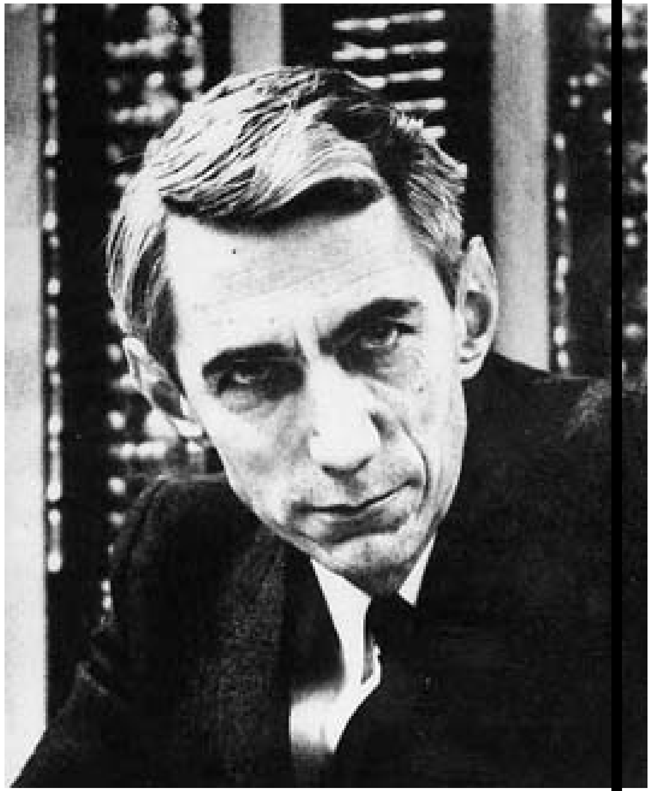
- There are at most $((n+5) \times m^2)^m$ boolean circuits with m or fewer gates (see next page).
- But $((n+5) \times m^2)^m < 2^{2^n}$ when $m = 2^n/(2n)$:

$$\begin{aligned} & m \log_2((n+5) \times m^2) \\ = \; & 2^n \left(1 - \frac{\log_2 \frac{4n^2}{n+5}}{2n} \right) \\ < \; & 2^n \end{aligned}$$

for $n \geq 2$.


n+5 choices
m choices
m choices

Claude Elwood Shannon (1916–2001)



Howard Gardner, "[Shannon's master's thesis is] possibly the most important, and also the most famous, master's thesis of the century."

Comments

- The lower bound $2^n/(2n)$ is rather tight because an upper bound is $n2^n$ (p. 186).
- The proof counted the number of circuits.
  - Some circuits may not be valid at all.
  - Different circuits may also compute the same function.
- Both are fine because we only need an upper bound on the number of circuits.
- We do not need to consider the outdoing edges because they have been counted as incoming edges.

Relations between Complexity Classes

It is, I own, not uncommon to be wrong in theory
and right in practice.
— Edmund Burke (1729–1797),
A Philosophical Enquiry into the Origin of Our Ideas of the Sublime and Beautiful (1757)

Proper (Complexity) Functions

- We say that $f : \mathbb{N} \to \mathbb{N}$ is a **proper (complexity) function** if the following hold:
  - f is nondecreasing.
  - There is a k-string TM M_f such that $M_f(x) = \sqcap^{f(|x|)}$ for any x.[a]
  - M_f halts after $O(|x| + f(|x|))$ steps.
  - M_f uses $O(f(|x|))$ space besides its input x.
- M_f's behavior depends only on $|x|$ not x's contents.
- M_f's running time is bounded by $f(n)$.

[a]The textbook calls "$\sqcap$" the quasi-blank symbol. The use of $M_f(x)$ will become clear in Proposition 16 (p. 204).

Examples of Proper Functions

- Most "reasonable" functions are proper: c, $\lceil \log n \rceil$, polynomials of n, 2^n, $\sqrt{n}$, $n!$, etc.
- If f and g are proper, then so are $f + g$, fg, and 2^g.[a]
- Nonproper functions when serving as the time bounds for complexity classes spoil "the theory building."
  - For example, $\text{TIME}(f(n)) = \text{TIME}(2^{f(n)})$ for some recursive function f (the **gap theorem**).[b]
- Only proper functions f will be used in $\text{TIME}(f(n))$, $\text{SPACE}(f(n))$, $\text{NTIME}(f(n))$, and $\text{NSPACE}(f(n))$.

[a]For $f(g)$, we need to add $f(n) \geq n$.
[b]Trakhtenbrot (1964); Borodin (1972).

Precise Turing Machines

- A TM M is **precise** if there are functions f and g such that for every $n \in \mathbb{N}$, for every x of length n, and for every computation path of M,
  - M halts after precisely $f(n)$ steps, and
  - All of its strings are of length precisely $g(n)$ at halting.
    - $*$ Recall that if M is a TM with input and output, we exclude the first and last strings.
- M can be deterministic or nondeterministic.

Precise TMs Are General

Proposition 16 *Suppose a TM*[a] M *decides* L *within time (space)* $f(n)$*, where* f *is proper. Then there is a precise TM* M' *which decides* L *in time* $O(n + f(n))$ *(space* $O(f(n))$*, respectively).*

- M' on input x first simulates the TM M_f associated with the proper function f on x.
- M_f's output of length $f(|\,x\,|)$ will serve as a "yardstick" or an "alarm clock."
- $M'(x)$ halts when and only when the alarm clock runs out—even if M halts earlier.

[a]It can be deterministic or nondeterministic.

The Proof (continued)

- If f is a time bound:
  - The simulation of each step of M on x is matched by advancing the cursor on the "clock" string.
  - M' stops at the moment the "clock" string is exhausted—even if $M(x)$ stops before that time.
  - So it is precise.
  - The time bound is therefore $O(|\,x\,| + f(|\,x\,|))$.

The Proof (concluded)

- If f is a space bound:
  - M' simulates M *on* the quasi-blanks of M_f's output string.
  - As before, M' stops at the moment the "clock" string is exhausted—even if $M(x)$ stops before that time.
  - So it is again precise.
  - The total space, not counting the input string, is $O(f(n))$.

Important Complexity Classes

- We write expressions like n^k to denote the union of all complexity classes, one for each value of k.
- For example,

$$\text{NTIME}(n^k) = \bigcup_{j>0} \text{NTIME}(n^j).$$

Important Complexity Classes (concluded)

$$\begin{aligned}
\text{P} &= \text{TIME}(n^k), \\
\text{NP} &= \text{NTIME}(n^k), \\
\text{PSPACE} &= \text{SPACE}(n^k), \\
\text{NPSPACE} &= \text{NSPACE}(n^k), \\
\text{E} &= \text{TIME}(2^{kn}), \\
\text{EXP} &= \text{TIME}(2^{n^k}), \\
\text{L} &= \text{SPACE}(\log n), \\
\text{NL} &= \text{NSPACE}(\log n).
\end{aligned}$$

Complements of Nondeterministic Classes

- R, RE, and coRE are distinct (p. 155).
  - coRE contains the complements of languages in RE, *not* the languages not in RE.
- Recall that the **complement** of L, denoted by $\bar{L}$, is the language $\Sigma^* - L$.
  - SAT COMPLEMENT is the set of unsatisfiable boolean expressions.

The Co-Classes

- For any complexity class $\mathcal{C}$, co$\mathcal{C}$ denotes the class

$$\{L : \bar{L} \in \mathcal{C}\}.$$

- Clearly, if $\mathcal{C}$ is a *deterministic* time or space *complexity class*, then $\mathcal{C} = \text{co}\mathcal{C}$.
  - They are said to be **closed under complement**.
  - A deterministic TM deciding L can be converted to one that decides $\bar{L}$ within the same time or space bound by reversing the "yes" and "no" states (p. 152).
- Whether nondeterministic classes for time are closed under complement is not known (p. 96).

Comments

- As
$$\text{co}\mathcal{C} = \{L : \bar{L} \in \mathcal{C}\},$$
$L \in \mathcal{C}$ if and only if $\bar{L} \in \text{co}\mathcal{C}$.
- But it is *not* true that $L \in \mathcal{C}$ if and only if $L \notin \text{co}\mathcal{C}$.
  - co$\mathcal{C}$ is not defined as $\bar{\mathcal{C}}$.
- For example, suppose $\mathcal{C} = \{\{2, 4, 6, 8, 10, \ldots\}\}$.
- Then co$\mathcal{C} = \{\{1, 3, 5, 7, 9, \ldots\}\}$.
- But $\bar{\mathcal{C}} = 2^{\{1,2,3,\ldots\}^*} - \{\{2, 4, 6, 8, 10, \ldots\}\}$.

The Quantified Halting Problem

- Let $f(n) \geq n$ be proper.
- Define

$$H_f = \{M; x : M \text{ accepts input } x \text{ after at most } f(\,|\,x\,|\,) \text{ steps}\},$$

where M is deterministic.

- Assume the input is binary.

$H_f \in \mathsf{TIME}(f(n)^3)$

- For each input $M; x$, we simulate M on x with an alarm clock of length $f(|\,x\,|)$.
  - Use the single-string simulator (p. 72), the universal TM (p. 137), and the linear speedup theorem (p. 81).
  - Our simulator accepts $M; x$ if and only if M accepts x before the alarm clock runs out.
- From p. 79, the total running time is $O(\ell_M k_M^2 f(n)^2)$, where ℓ_M is the length to encode each symbol or state of M and k_M is M's number of strings.
- As $\ell_M k_M^2 = O(n)$, the running time is $O(f(n)^3)$, where the constant is independent of M.

$H_f \notin \mathsf{TIME}(f(\lfloor n/2 \rfloor))$

- Suppose TM M_{H_f} decides H_f in time $f(\lfloor n/2 \rfloor)$.
- Consider machine:

```
$D_f(M)$   {
        **if** $M_{H_f}(M;M) =$ "yes"
        **then** "no";
        **else** "yes";
}
```

- D_f on input M runs in the same time as M_{H_f} on input $M;M$, i.e., in time $f(\lfloor \frac{2n+1}{2} \rfloor) = f(n)$, where $n = |\, M \,|$.[a]

[a] A student pointed out on October 6, 2004, that this estimation omits the time to write down $M;M$.

The Proof (concluded)

- First,

$$\begin{aligned} & D_f(D_f) = \text{“yes”} \\ \Rightarrow\quad & D_f; D_f \notin H_f \\ \Rightarrow\quad & D_f \text{ does not accept } D_f \text{ within time } f(\mid D_f \mid) \\ \Rightarrow\quad & D_f(D_f) \neq \text{“yes”} \\ \Rightarrow\quad & D_f(D_f) = \text{“no”} \end{aligned}$$

  a contradiction

- Similarly, $D_f(D_f) =$ “no” $\Rightarrow D_f(D_f) =$ “yes.”

The Time Hierarchy Theorem

Theorem 17 *If $f(n) \geq n$ is proper, then*

$$\text{TIME}(f(n)) \subsetneq \text{TIME}(f(2n+1)^3).$$

- The quantified halting problem makes it so.

Corollary 18 P $\subsetneq$ E.

- P $\subseteq$ TIME(2^n) because poly$(n) \leq 2^n$ for n large enough.
- But by Theorem 17,

$$\text{TIME}(2^n) \subsetneq \text{TIME}((2^{2n+1})^3) \subseteq \text{E}.$$

- So P $\subsetneq$ E.

The Space Hierarchy Theorem

Theorem 19 (Hennie and Stearns (1966)) *If $f(n)$ is proper, then*

$$\mathrm{SPACE}(f(n)) \subsetneq \mathrm{SPACE}(f(n) \log f(n)).$$

Corollary 20 $\mathrm{L} \subsetneq \mathrm{PSPACE}$.

Nondeterministic Time Hierarchy Theorems

Theorem 21 (Cook (1973)) $\text{NTIME}(n^r) \subsetneq \text{NTIME}(n^s)$ *whenever* $1 \leq r < s$.

Theorem 22 (Seiferas, Fischer, and Meyer (1978)) *If* $T_1(n), T_2(n)$ *are proper, then*

$$\text{NTIME}(T_1(n)) \subsetneq \text{NTIME}(T_2(n))$$

whenever $T_1(n+1) = o(T_2(n))$.