Theory of Computation

Homework 3 Solution

Problem 1. Define D-SAT = $\{\Theta \mid \Theta$ is a Boolean expression with at least two satisfying assignments$\}$. Show that D-SAT is NP-complete. (Do not forget to show it is in NP.)

Proof. We nondeterministically generate two different assignments and verify that every clause is satisfied by both cases. Thus, D-SAT $\in$ NP.

We reduce the 3SAT to D-SAT. Given a Boolean expression Φ, we create a new Boolean expression Φ' by inserting a new clause $(a' \vee a' \vee \overline{a'})$ to Φ, where the new variable a' does not appear in Φ. If $\Phi \in$ 3SAT, then we have at least two satisfying assignments by setting $a' = 1$ and $a' = 0$ for Φ' respectively to any assignment that satisfies Φ. For another direction, if Φ' is satisfiable with at least two assignments, then there must exist a truth assignment satisfying the original expression Φ, therefore $\Phi \in$ 3SAT. The reduction runs clearly in polynomial time. Hence, D-SAT is NP-complete.

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Problem 2. We define "SE-Hamiltonian path" as a path that visits all the nodes once in an undirected graph which starts from a node n_s and ends at a node n_e in the graph, where both n_s and n_e are inputs. Show that SE-Hamiltonian path is NP-complete. (Hint: Hamiltonian cycle is NP-complete. Do not forget to show it is in NP.)

Proof. By traversing a nondeterministically generated path, we verify that the path visits all the nodes exactly once, starting from n_s and ending in n_e. Hence, SE-Hamiltonian Path ∈ NP.

We next reduce Hamiltonian cycle to SE-Hamiltonian path. Suppose we are given an undirected graph $G(V, E)$, where V is the set of nodes in G and E is the set of edges in G. Let v be node 1 in G. Add a new node v' to G and create a new undirected graph $G'(V', E')$, where $V' = V \cup \{v'\}$ and $E' = E \cup \{(u, v') | (u, v) \in E\}$. So the new node v' in G' connects to exactly the same nodes as v and can be viewed as a copy of v. Now, set $n_s = v$ and $n_e = v'$. Suppose there is a Hamiltonian cycle $(v, v_1, v_2, \ldots, v_{n-1}, v)$ in G. Then $(v, v_1, v_2, \ldots, v_{n-1}, v')$ is a Hamiltonian path from v to v' in G'. For another direction, suppose G' has a Hamiltonian path $(v, v_1, v_2, \ldots, v_{n-1}, v')$ from v to v', then $(v, v_1, v_2, \ldots, v_{n-1}, v)$ is a Hamiltonian cycle in G. The reduction is clearly doable in polynomial time. Thus SE-Hamiltonian path is NP-complete.

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