Simulating Nondeterministic TMs

Nondeterminism does not add power to TMs.

Theorem 4 *Suppose language L is decided by an NTM N in time $f(n)$. Then it is decided by a 3-string deterministic TM M in time $O(c^{f(n)})$, where $c > 1$ is some constant depending on N.*

- On input x, M goes down every computation path of N using depth-first search.
  - M does *not* need to know $f(n)$.
  - As N is time-bounded, the depth-first search will not run indefinitely.

The Proof (concluded)

- If any path leads to "yes," then M immediately enters the "yes" state.
- If none of the paths leads to "yes," then M enters the "no" state.
- The simulation takes time $O(c^{f(n)})$ for some $c > 1$ because the computation tree has that many nodes.

Corollary 5 $\text{NTIME}(f(n))) \subseteq \bigcup_{c>1} \text{TIME}(c^{f(n)})$.

NTIME vs. TIME

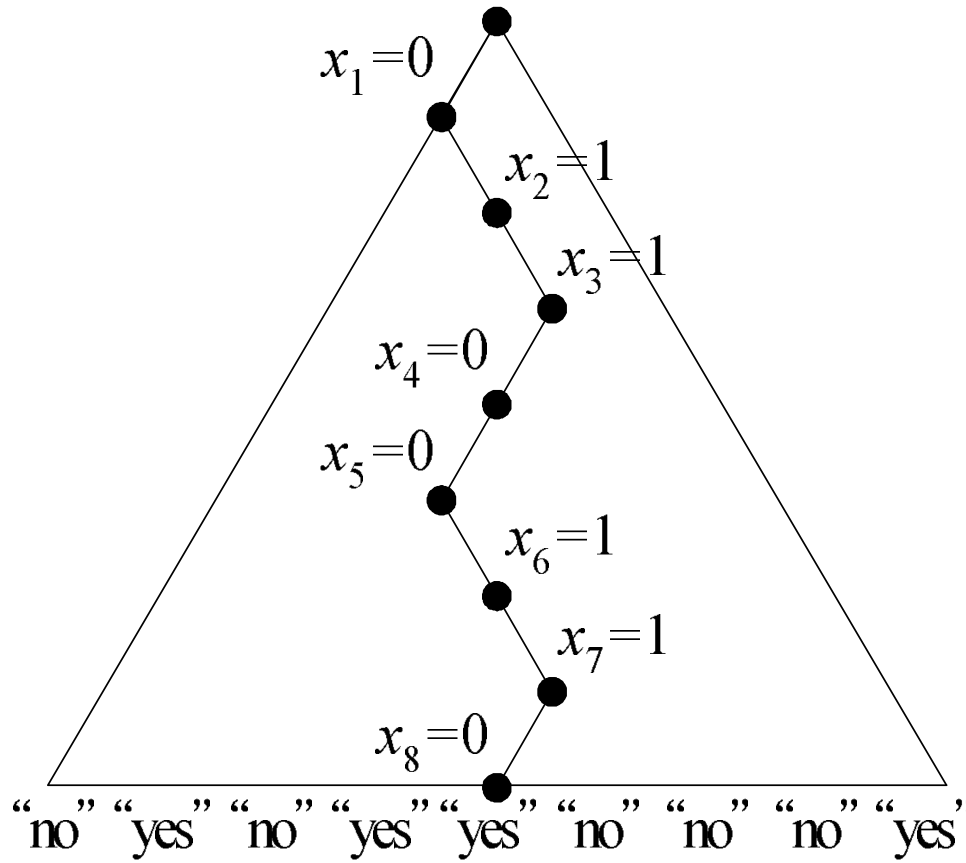
- Does converting an NTM into a TM require exploring all computation paths of the NTM as done in Theorem 4 (p. 97)?
- This is the most important question in theory with important practical implications.

A Nondeterministic Algorithm for Satisfiability

ϕ is a boolean formula with n variables.

for $i = 1, 2, \ldots, n$ **do**
    Guess $x_i \in \{0, 1\}$; {Nondeterministic choice.}
end for
{Verification:}
if $\phi(x_1, x_2, \ldots, x_n) = 1$ **then**
    "yes";
else
    "no";
end if

Computation Tree for Satisfiability



Analysis

- The computation tree is a complete binary tree of depth n.
- Every computation path corresponds to a particular truth assignment out of 2^n.
- ϕ is satisfiable iff there is a truth assignment that satisfies ϕ.

Analysis (concluded)

- The algorithm decides language $\{\phi : \phi \text{ is satisfiable}\}$.
  - Suppose ϕ is satisfiable.
  - That means there is a truth assignment that satisfies ϕ.
  - So there is a computation path that results in "yes."
  - Suppose ϕ is not satisfiable.
  - That means every truth assignment makes ϕ false.
  - So every computation path results in "no."
- General paradigm: Guess a "proof" and then verify it.

The Traveling Salesman Problem

- We are given n cities $1, 2, \ldots, n$ and integer distance d_{ij} between any two cities i and j.
- Assume $d_{ij} = d_{ji}$ for convenience.
- The **traveling salesman problem** (TSP) asks for the total distance of the shortest tour of the cities.
- The decision version TSP (D) asks if there is a tour with a total distance at most B, where B is an input.[a]

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[a]Both problems are extremely important and are equally hard (p. 354 and p. 447).

A Nondeterministic Algorithm for TSP (D)

for $i = 1, 2, \ldots, n$ **do**
    Guess $x_i \in \{1, 2, \ldots, n\}$; {The ith city.}[a]
end for
$x_{n+1} := x_1$;
{Verification:}
if $x_1, x_2, \ldots, x_n$ are distinct and $\sum_{i=1}^{n} d_{x_i, x_{i+1}} \leq B$ **then**
    "yes";
else
    "no";
end if

---

[a] Can be made into a series of $\log_2 n$ *binary* choices for each x_i so that the next-state count (2) is a constant, independent of input size. Contributed by Mr. Chih-Duo Hong (R95922079) on September 27, 2006.

Analysis

- Suppose the input graph contains at least one tour of the cities with a total distance at most B.
- Then there is a computation path that leads to "yes."[a]
- Suppose the input graph contains no tour of the cities with a total distance at most B.
- Then every computation path leads to "no."

[a] It does not mean the algorithm will follow that path. It just means such a computation path exists.

Remarks on the $\mathrm{P} \stackrel{?}{=} \mathrm{NP}$ Open Problem[a]

- Many practical applications depend on answers to the $\mathrm{P} \stackrel{?}{=} \mathrm{NP}$ question.
- Verification of password is easy (so it is in NP).
  - A computer should not take a long time to let a user log in.
- A password system should be hard to crack (loosely speaking, cracking it should not be in P).

[a]Contributed by Mr. Kuan-Lin Huang (B96902079, R00922018) on September 27, 2011.

Nondeterministic Space Complexity Classes

- Let L be a language.
- Then

$$L \in \text{NSPACE}(f(n))$$

  if there is an NTM with input and output that decides L and operates within space bound $f(n)$.
- NSPACE($f(n)$) is a set of languages.
- As in the linear speedup theorem (Theorem 3 on p. 75), constant coefficients do not matter.

Graph Reachability

- Let $G(V, E)$ be a directed graph (**digraph**).
- REACHABILITY asks if, given nodes a and b, does G contain a path from a to b?
- Can be easily solved in polynomial time by breadth-first search.
- How about its nondeterministic space complexity?

The First Try: NSPACE($n \log n$)

$x_1 := a$; {Assume $a \neq b$.}
for $i = 2, 3, \ldots, n$ **do**
    Guess $x_i \in \{v_1, v_2, \ldots, v_n\}$; {The ith node.}
end for
for $i = 2, 3, \ldots, n$ **do**
    if $(x_{i-1}, x_i) \notin E$ **then**
        "no";
    end if
    if $x_i = b$ **then**
        "yes";
    end if
end for
"no";

In Fact, REACHABILITY $\in$ NSPACE$(\log n)$

$x := a$;
for $i = 2, 3, \ldots, n$ **do**
  Guess $y \in \{v_1, v_2, \ldots, v_n\}$; {The next node.}
  if $(x, y) \notin E$ **then**
    "no";
  end if
  if $y = b$ **then**
    "yes";
  end if
  $x := y$;
end for
"no";

Space Analysis

- Variables i, x, and y each require $O(\log n)$ bits.
- Testing $(x, y) \in E$ is accomplished by consulting the input string with counters of $O(\log n)$ bits long.
- Hence

$$\textsc{reachability} \in \text{NSPACE}(\log n).$$

  - REACHABILITY with more than one terminal node also has the same complexity.

- REACHABILITY $\in$ P (p. 211).

Undecidability

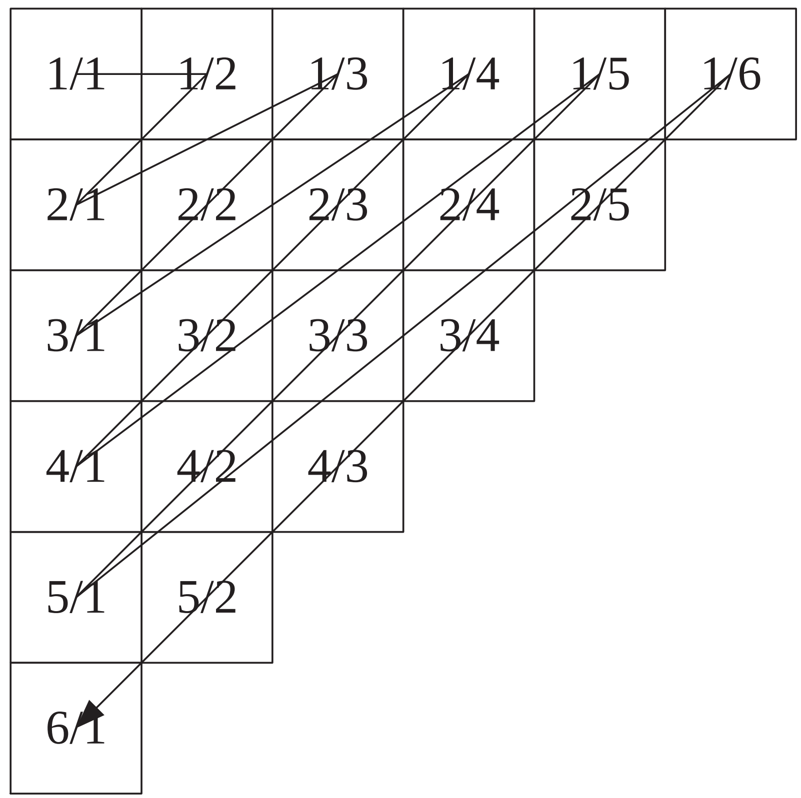
God exists since mathematics is consistent,
and the Devil exists since we cannot prove it.
— André Weil (1906–1998)

Whatsoever we imagine is *finite*.
Therefore there is no idea, or conception
of any thing we call *infinite*.
— Thomas Hobbes (1588–1679), *Leviathan*

Infinite Sets

- A set is **countable** if it is finite or if it can be put in one-one correspondence with $\mathbb{N} = \{ 0, 1, \ldots \}$, the set of natural numbers.
  - Set of integers $\mathbb{Z}$.
    * $0 \leftrightarrow 0$.
    * $1 \leftrightarrow 1, 2 \leftrightarrow 3, 3 \leftrightarrow 5, \ldots$.
    * $-1 \leftrightarrow 2, -2 \leftrightarrow 4, -3 \leftrightarrow 6, \ldots$.
  - Set of positive integers $\mathbb{Z}^+$: $i \leftrightarrow i - 1$.
  - Set of positive odd integers: $i \leftrightarrow (i - 1)/2$.
  - Set of (positive) rational numbers: See next page.
  - Set of squared integers: $i \leftrightarrow \sqrt{i}$.

Rational Numbers Are Countable



Cardinality

- For any set A, define $|A|$ as A's **cardinality** (size).
- Two sets are said to have the same cardinality, or

$$|A| = |B| \quad \text{or} \quad A \sim B,$$

  if there exists a one-to-one correspondence between their elements.
- 2^A denotes set A's **power set**, that is $\{B : B \subseteq A\}$.
  - E.g., $\{0, 1\}$'s power set is $2^{\{0,1\}} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$.
- If $|A| = k$, then $|2^A| = 2^k$.

Cardinality (concluded)

- Define $|A| \leq |B|$ if there is a one-to-one correspondence between A and a subset of B's.
- Obviously, if $A \subseteq B$, then $|A| \leq |B|$.
  - So $|\,\mathbb{N}\,| \leq |\,\mathbb{Z}\,|$.
  - So $|\,\mathbb{N}\,| \leq |\,\mathbb{R}\,|$.
- Define $|A| < |B|$ if $|A| \leq |B|$ but $|A| \neq |B|$.
- If $A \subsetneq B$, then $|A| < |B|$?

Cardinality and Infinite Sets

- If A and B are infinite sets, it is possible that $A \subsetneq B$ yet $|A| = |B|$.
  - The set of integers *properly* contains the set of odd integers.
  - But the set of integers has the same cardinality as the set of odd integers (p. 115).[a]
- A lot of "paradoxes."

[a]Leibniz uses it to "prove" that there are no infinite numbers (Russell, 1914).

Galileo's[a] Paradox (1638)

- The squares of the positive integers can be placed in one-to-one correspondence with all the positive integers.
- This is contrary to the axiom of Euclid[b] that the whole is greater than any of its proper parts.[c]
- Resolution of paradoxes: Pick the notion that results in "better" mathematics.
- The difference between a mathematical paradox and a contradiction is often a matter of opinions.

---

[a] Galileo (1564–1642).
[b] Euclid (325 B.C.–265 B.C.).
[c] Leibniz never challenges that axiom (Knobloch, 1999).

Hilbert's[a] Paradox of the Grand Hotel

- For a hotel with a finite number of rooms with all the rooms occupied, a new guest will be turned away.
- Now imagine a hotel with an infinite number of rooms, all of which are occupied.
- A new guest comes and asks for a room.
- "But of course!" exclaims the proprietor.
- He moves the person previously occupying Room 1 to Room 2, the person from Room 2 to Room 3, and so on.
- The new customer now occupies Room 1.

[a]David Hilbert (1862–1943).

Hilbert's Paradox of the Grand Hotel (concluded)

- Now imagine a hotel with an infinite number of rooms, all taken up.
- An infinite number of new guests come in and ask for rooms.
- "Certainly," says the proprietor.
- He moves the occupant of Room 1 to Room 2, the occupant of Room 2 to Room 4, and so on.
- Now all odd-numbered rooms become free and the infinity of new guests can be accommodated in them.
- "There are many rooms in my Father's house, and I am going to prepare a place for you." (*John* 14:3)

David Hilbert (1862–1943)

Cantor's Theorem

Theorem 6 *The set of all subsets of* $\mathbb{N}$ $(2^{\mathbb{N}})$ *is infinite and not countable.*

- Suppose $(2^{\mathbb{N}})$ is countable with $f : \mathbb{N} \to 2^{\mathbb{N}}$ being a bijection.[a]
- Consider the set $B = \{k \in \mathbb{N} : k \notin f(k)\} \subseteq \mathbb{N}$.
- Suppose $B = f(n)$ for some $n \in \mathbb{N}$.

---

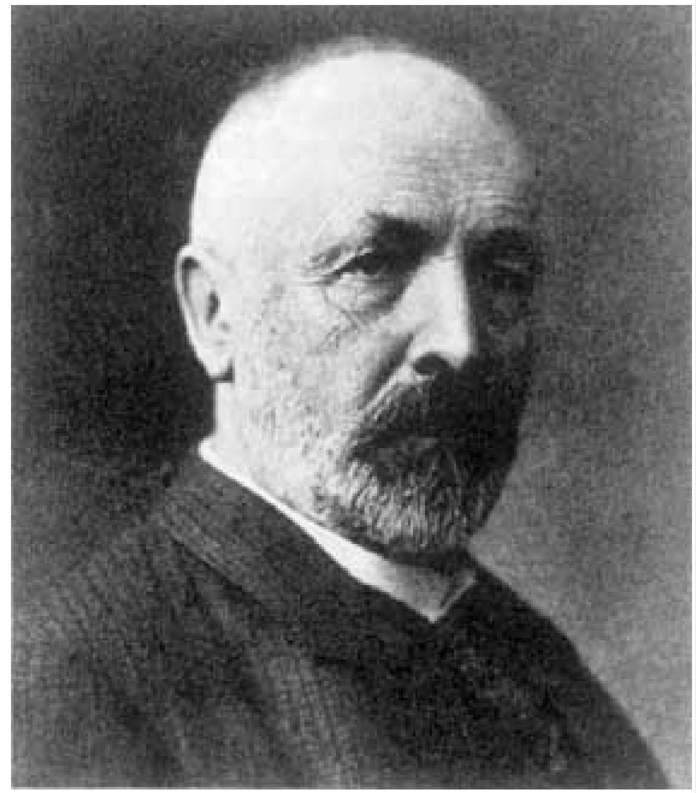
[a]Note that $f(k)$ is a subset of $\mathbb{N}$.

The Proof (concluded)

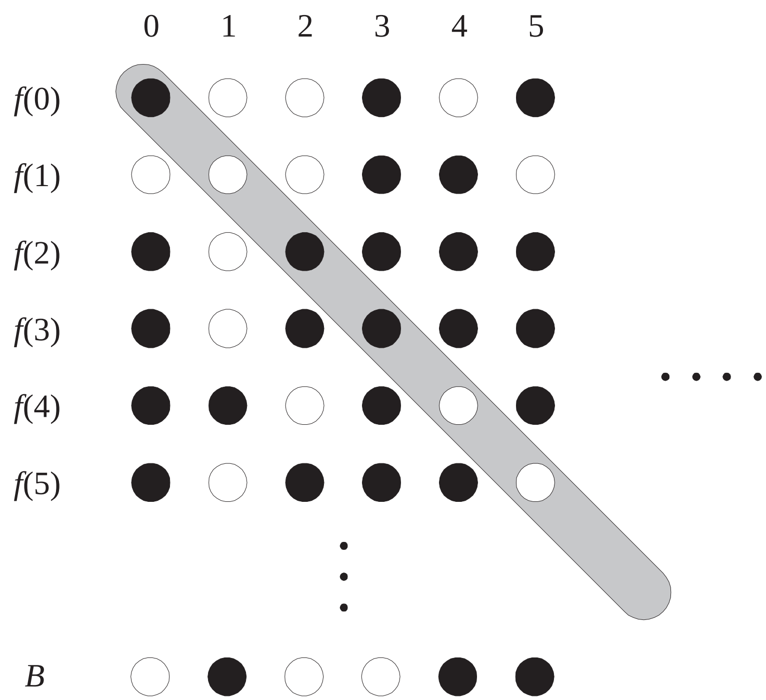
- If $n \in f(n) = B$, then $n \in B$, but then $n \notin B$ by B's definition.
- If $n \notin f(n) = B$, then $n \notin B$, but then $n \in B$ by B's definition.
- Hence $B \neq f(n)$ for any n.
- f is not a bijection, a contradiction.

Georg Cantor (1845–1918)

Kac and Ulam (1968), "[If] one had to name a single person whose work has had the most decisive influence on the present spirit of mathematics, it would almost surely be Georg Cantor."



Cantor's Diagonalization Argument Illustrated



A Corollary of Cantor's Theorem

Corollary 7 *For any set T, finite or infinite,*

$$|\,T\,| < |\,2^T\,|.$$

- The inequality holds in the finite T case as $k < 2^k$.
- Assume T is infinite now.
- To prove $|\,T\,| \leq |2^T|$, simply consider $f(x) = \{x\} \in 2^T$.
  - f maps a member of $T = \{\, a, b, c, \ldots \,\}$ to a corresponding member of $\{\, \{\, a \,\}, \{\, b \,\}, \{\, c \,\}, \ldots \,\} \subseteq 2^T$.
- To prove the strict inequality $|\,T\,| \lneq |2^T|$, we use the same argument as Cantor's theorem.

A Second Corollary of Cantor's Theorem

Corollary 8 *The set of all functions on* $\mathbb{N}$ *is not countable.*

- It suffices to prove it for functions from $\mathbb{N}$ to $\{0, 1\}$.
- Every function $f : \mathbb{N} \to \{0, 1\}$ determines a subset of $\mathbb{N}$:

  $$\{n : f(n) = 1\} \subseteq \mathbb{N},$$

  and vice versa.
- So the set of functions from $\mathbb{N}$ to $\{0, 1\}$ has cardinality $|\, 2^{\mathbb{N}} \,|$.
- Cantor's theorem (p. 124) then implies the claim.

Existence of Uncomputable Problems

- Every program is a finite sequence of 0s and 1s, thus a nonnegative integer.[a]
- Hence every program corresponds to some integer.
- The set of programs is countable.

---

[a]Different binary strings may be mapped to the same integer (e.g., "001" and "01"). To prevent it, use the lexicographic order as the mapping or simply insert "1" as the most significant bit of the binary string before the mapping (so "001" becomes "1001"). Contributed by Mr. Yu-Chih Tung (R98922167) on October 5, 2010.

Existence of Uncomputable Problems (concluded)

- A function is a mapping from integers to integers.
- The set of functions is not countable by Corollary 8 (p. 129).
- So there are functions for which no programs exist.[a]

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[a] As a nondeterministic program may not compute a function, we consider only deterministic programs for this sentence. Contributed by Mr. Patrick Will (`A99725101`) on October 5, 2010.

Universal Turing Machine[a]

- A **universal Turing machine** U interprets the input as the *description* of a TM M concatenated with the *description* of an input to that machine, x.
  - Both M and x are over the alphabet of U.
- U simulates M on x so that

$$U(M; x) = M(x).$$

- U is like a modern computer, which executes any valid machine code, or a Java Virtual machine, which executes any valid bytecode.

[a] Turing (1936).

The Halting Problem

- **Undecidable problems** are problems that have no algorithms.
- Equivalently, they are languages that are not recursive.
- We knew undecidable problems exist (p. 130).
- We now define a concrete undecidable problem, the **halting problem**:

$$H = \{M; x : M(x) \neq \nearrow\}.$$

  – Does M halt on input x?

H Is Recursively Enumerable

- Use the universal TM U to simulate M on x.
- When M is about to halt, U enters a "yes" state.
- If $M(x)$ diverges, so does U.
- This TM accepts H.