The Secret-Key Agreement Problem

- Exchanging messages securely using a private-key cryptosystem requires Alice and Bob possessing the same key (p. 561).
- How can they agree on the same secret key when the channel is insecure?
- This is called the **secret-key agreement problem**.
- It was solved by Diffie and Hellman (1976) using one-way functions.

The Diffie-Hellman Secret-Key Agreement Protocol

1: Alice and Bob agree on a large prime p and a primitive root g of p; $\{p$ and g are public.$\}$

2: Alice chooses a large number a at random;

3: Alice computes $\alpha = g^a \bmod p$;

4: Bob chooses a large number b at random;

5: Bob computes $\beta = g^b \bmod p$;

6: Alice sends α to Bob, and Bob sends β to Alice;

7: Alice computes her key $\beta^a \bmod p$;

8: Bob computes his key $\alpha^b \bmod p$;

Analysis

- The keys computed by Alice and Bob are identical:

$$\beta^a = g^{ba} = g^{ab} = \alpha^b \bmod p.$$

- To compute the common key from p, g, α, β is known as the **Diffie-Hellman problem**.
- It is conjectured to be hard.
- If discrete logarithm is easy, then one can solve the Diffie-Hellman problem.
  - Because a and b can then be obtained by Eve.
- But the other direction is still open.

A Parallel History

- Diffie and Hellman's solution to the secret-key agreement problem led to public-key cryptography.
- At around the same time (or earlier) in Britain, the RSA public-key cryptosystem was invented first before the Diffie-Hellman secret-key agreement scheme was.
  - Ellis, Cocks, and Williamson of the Communications Electronics Security Group of the British Government Communications Head Quarters (GCHQ).

Digital Signatures[a]

- Alice wants to send Bob a *signed* document x.
- The signature must unmistakably identifies the sender.
- Both Alice and Bob have public and private keys

$$e_{\text{Alice}}, e_{\text{Bob}}, d_{\text{Alice}}, d_{\text{Bob}}.$$

- Every cryptosystem guarantees $D(d, E(e, x)) = x$.
- Assume the cryptosystem also satisfies the commutative property

$$E(e, D(d, x)) = D(d, E(e, x)). \tag{9}$$

  – As $(x^d)^e = (x^e)^d$, the RSA system satisfies it.

[a]Diffie and Hellman (1976).

Digital Signatures Based on Public-Key Systems

- Alice signs x as
$$(x, D(d_{\text{Alice}}, x)).$$
- Bob receives (x, y) and verifies the signature by checking
$$E(e_{\text{Alice}}, y) = E(e_{\text{Alice}}, D(d_{\text{Alice}}, x)) = x$$
based on Eq. (9).
- The claim of authenticity is founded on the difficulty of inverting E_{Alice} without knowing the key d_{Alice}.

Probabilistic Encryption[a]

- A deterministic cryptosystem can be broken if the plaintext has a distribution that favors the "easy" cases.
- The ability to forge signatures on even a vanishingly small fraction of strings of some length is a security weakness if those strings were the probable ones!
- A scheme may also "leak" *partial* information.
  - Parity of the plaintext, e.g.
- The first solution to the problems of skewed distribution and partial information was based on the QRA.

[a]Goldwasser and Micali (1982).

Shafi Goldwasser (1958–)

Silvio Micali (1954–)

A Useful Lemma

Lemma 75 *Let $n = pq$ be a product of two distinct primes. Then a number $y \in Z_n^*$ is a quadratic residue modulo n if and only if $(y \,|\, p) = (y \,|\, q) = 1$.*

- The "only if" part:
  - Let x be a solution to $x^2 = y \bmod pq$.
  - Then $x^2 = y \bmod p$ and $x^2 = y \bmod q$ also hold.
  - Hence y is a quadratic modulo p and a quadratic residue modulo q.

The Proof (concluded)

- The "if" part:
  - Let $a_1^2 = y \bmod p$ and $a_2^2 = y \bmod q$.
  - Solve

$$\begin{aligned} x &= a_1 \bmod p, \\ x &= a_2 \bmod q, \end{aligned}$$

    for x with the Chinese remainder theorem.
  - As $x^2 = y \bmod p$, $x^2 = y \bmod q$, and $\gcd(p, q) = 1$, we must have $x^2 = y \bmod pq$.

The Jacobi Symbol and Quadratic Residuacity Test

- The Legendre symbol can be used as a test for quadratic residuacity by Lemma 63 (p. 480).
- Lemma 75 (p. 586) says this is not the case with the Jacobi symbol in general.
- Suppose $n = pq$ is a product of two distinct primes.
- A number $y \in Z_n^*$ with Jacobi symbol $(y \mid pq) = 1$ may be a quadratic nonresidue modulo n when

$$(y \mid p) = (y \mid q) = -1,$$

because $(y \mid pq) = (y \mid p)(y \mid q)$.

The Setup

- Bob publishes $n = pq$, a product of two distinct primes, and a quadratic nonresidue y with Jacobi symbol 1.
- Bob keeps secret the factorization of n.
- Alice wants to send bit string $b_1 b_2 \cdots b_k$ to Bob.
- Alice encrypts the bits by choosing a random quadratic residue modulo n if b_i is 1 and a random quadratic nonresidue (with Jacobi symbol 1) otherwise.
- A sequence of residues and nonresidues are sent.
- Knowing the factorization of n, Bob can efficiently test quadratic residuacity and thus read the message.

The Protocol for Alice

for $i = 1, 2, \ldots, k$ **do**
  Pick $r \in Z_n^*$ randomly;
  if $b_i = 1$ **then**
    Send $r^2 \bmod n$; {Jacobi symbol is 1.}
  else
    Send $r^2 y \bmod n$; {Jacobi symbol is still 1.}
  end if
end for

The Protocol for Bob

for $i = 1, 2, \ldots, k$ **do**
    Receive r;
    if $(r \,|\, p) = 1$ and $(r \,|\, q) = 1$ **then**
        $b_i := 1$;
    else
        $b_i := 0$;
    end if
end for

Semantic Security

- This encryption scheme is probabilistic.
- There are a large number of different encryptions of a given message.
- One is chosen at random by the sender to represent the message.
- This scheme is both polynomially secure and **semantically secure**.

What Is a Proof?

- A proof convinces a party of a certain claim.
  - "$x^n + y^n \neq z^n$ for all $x, y, z \in \mathbb{Z}^+$ and $n > 2$."
  - "Graph G is Hamiltonian."
  - "$x^p = x \bmod p$ for prime p and $p \not| x$."
- In mathematics, a proof is a fixed sequence of theorems.
  - Think of it as a written examination.
- We will extend a proof to cover a proof *process* by which the validity of the assertion is established.
  - Recall a job interview or an oral examination.

Prover and Verifier

- There are two parties to a proof.
  - The **prover** (**Peggy**).
  - The **verifier** (**Victor**).
- Given an assertion, the prover's goal is to convince the verifier of its validity (**completeness**).
- The verifier's objective is to accept only correct assertions (**soundness**).
- The verifier usually has an easier job than the prover.
- The setup is very much like the Turing test.[a]

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[a] Turing (1950).

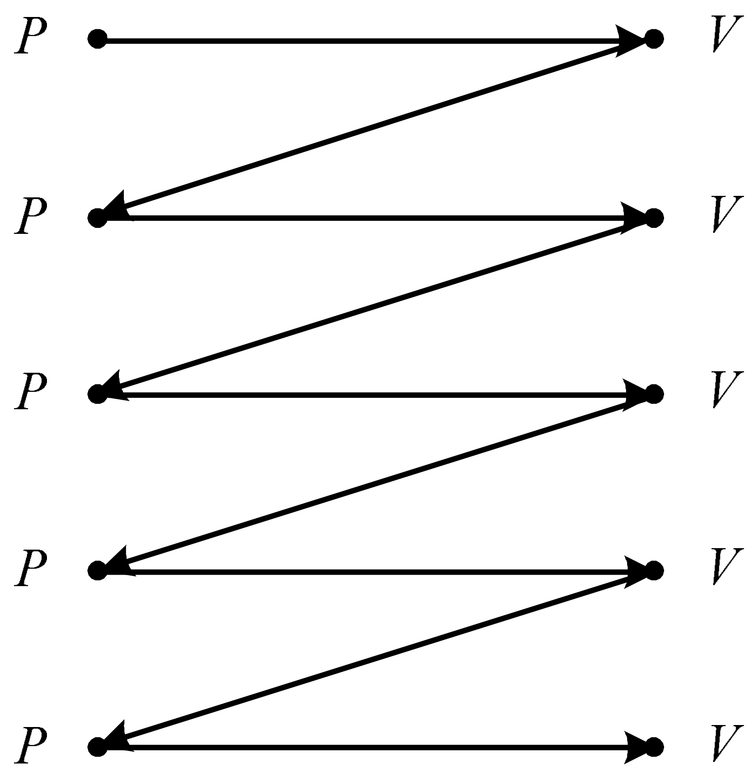
Interactive Proof Systems

- An **interactive proof** for a language L is a sequence of questions and answers between the two parties.
- At the end of the interaction, the verifier decides whether the claim is true or false.
- The verifier must be a probabilistic polynomial-time algorithm.
- The prover runs an exponential-time algorithm.
  – If the prover is not more powerful than the verifier, no interaction is needed.

Interactive Proof Systems (concluded)

- The system decides L if the following two conditions hold for any common input x.
  - If $x \in L$, then the probability that x is accepted by the verifier is at least $1 - 2^{-|x|}$.
  - If $x \notin L$, then the probability that x is accepted by the verifier with *any* prover replacing the original prover is at most $2^{-|x|}$.
- Neither the number of rounds nor the lengths of the messages can be more than a polynomial of $|x|$.

An Interactive Proof



IP[a]

- **IP** is the class of all languages decided by an interactive proof system.
- When $x \in L$, the completeness condition can be modified to require that the verifier accepts with certainty without affecting IP.[b]
- Similar things cannot be said of the soundness condition when $x \notin L$.
- Verifier's coin flips can be public.[c]

---

[a] Goldwasser, Micali, and Rackoff (1985).
[b] Goldreich, Mansour, and Sipser (1987).
[c] Goldwasser and Sipser (1989).

The Relations of IP with Other Classes

- NP $\subseteq$ IP.
  - IP becomes NP when the verifier is deterministic and there is only one round of interaction.
- BPP $\subseteq$ IP.
  - IP becomes BPP when the verifier ignores the prover's messages.
- IP actually coincides with PSPACE.[a]

[a] Shamir (1990).

Graph Isomorphism

- $V_1 = V_2 = \{1, 2, \ldots, n\}$.
- Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there exists a permutation π on $\{1, 2, \ldots, n\}$ so that $(u, v) \in E_1 \Leftrightarrow (\pi(u), \pi(v)) \in E_2$.
- The task is to answer if $G_1 \cong G_2$.
- No known polynomial-time algorithms.
- The problem is in NP (hence IP).
- It is not likely to be NP-complete.[a]

[a]Schöning (1987).

GRAPH NONISOMORPHISM

- $V_1 = V_2 = \{1, 2, \ldots, n\}$.
- Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **nonisomorphic** if there exist no permutations π on $\{1, 2, \ldots, n\}$ so that $(u, v) \in E_1 \Leftrightarrow (\pi(u), \pi(v)) \in E_2$.
- The task is to answer if $G_1 \not\cong G_2$.
- Again, no known polynomial-time algorithms.
  - It is in coNP, but how about NP or BPP?
  - It is not likely to be coNP-complete.
- Surprisingly, GRAPH NONISOMORPHISM $\in$ IP.[a]

[a] Goldreich, Micali, and Wigderson (1986).

A 2-Round Algorithm

Victor selects a random $i \in \{1, 2\}$;
Victor selects a random permutation π on $\{1, 2, \ldots, n\}$;
Victor applies π on graph G_i to obtain graph H;
Victor sends (G_1, H) to Peggy;
if $G_1 \cong H$ **then**
    Peggy sends $j = 1$ to Victor;
else
    Peggy sends $j = 2$ to Victor;
end if
if $j = i$ **then**
    Victor accepts;
else
    Victor rejects;
end if

Analysis

- Victor runs in probabilistic polynomial time.
- Suppose $G_1 \ncong G_2$.
  - Peggy is able to tell which G_i is isomorphic to H.
  - So Victor always accepts.
- Suppose $G_1 \cong G_2$.
  - No matter which i is picked by Victor, Peggy or any prover sees 2 identical graphs.
  - Peggy or any prover with exponential power has only probability one half of guessing i correctly.
  - So Victor erroneously accepts with probability 1/2.
- Repeat the algorithm to obtain the desired probabilities.

Knowledge in Proofs

- Suppose I know a satisfying assignment to a satisfiable boolean expression.
- I can convince Alice of this by giving her the assignment.
- But then I give her more knowledge than is necessary.
  - Alice can claim that she found the assignment!
  - Login authentication faces essentially the same issue.
  - See `www.wired.com/wired/archive/1.05/atm_pr.html` for a famous ATM fraud in the U.S.

Knowledge in Proofs (concluded)

- Suppose I always give Alice random bits.
- Alice extracts no knowledge from me by any measure, but I prove nothing.
- Question 1: Can we design a protocol to convince Alice (the knowledge) of a secret without revealing anything extra?
- Question 2: How to define this idea rigorously?

Zero Knowledge Proofs[a]

An interactive proof protocol (P, V) for language L has the **perfect zero-knowledge** property if:

- For every verifier V', there is an algorithm M with expected polynomial running time.
- M on any input $x \in L$ generates the same probability distribution as the one that can be observed on the communication channel of (P, V') on input x.

---

[a]Goldwasser, Micali, and Rackoff (1985).

Comments

- Zero knowledge is a property of the prover.
  - It is the robustness of the prover against attempts of the verifier to extract knowledge via interaction.
  - The verifier may deviate arbitrarily (but in polynomial time) from the predetermined program.
  - A verifier cannot use the transcript of the interaction to convince a third-party of the validity of the claim.
  - The proof is hence not transferable.

Comments (continued)

- Whatever a verifier can "learn" from the specified prover P via the communication channel could as well be computed from the verifier alone.
- The verifier does not learn anything except "$x \in L$."
- Zero-knowledge proofs yield no knowledge in the sense that they can be constructed by the verifier who believes the statement, and yet these proofs do convince him.

Comments (continued)

- The "paradox" is resolved by noting that it is not the transcript of the conversation that convinces the verifier.
- But the fact that this conversation was held "on line."
- *Computational* zero-knowledge proofs are based on complexity assumptions.
  - M only needs to generate a distribution that is computationally indistinguishable from the verifier's view of the interaction.

Comments (concluded)

- It is known that if one-way functions exist, then zero-knowledge proofs exist for every problem in NP.[a]
- The verifier can be restricted to the honest one (i.e., it follows the protocol).[b]
- The coins can be public.[c]

---

[a] Goldreich, Micali, and Wigderson (1986).
[b] Vadhan (2006).
[c] Vadhan (2006).

Are You Convinced?

- A newspaper commercial for hair-growing products for men.
  - A (for all practical purposes) bald man has a full head of hair after 3 months.
- A TV commercial for weight-loss products.
  - A (by any reasonable measure) overweight woman loses 10 kilograms in 10 weeks.

Quadratic Residuacity

- Let n be a product of two distinct primes.
- Assume extracting the square root of a quadratic residue modulo n is hard without knowing the factors.
- We next present a zero-knowledge proof for $x \in Z_n^*$ being a quadratic residue.

Zero-Knowledge Proof of Quadratic Residuacity

1: **for** $m = 1, 2, \ldots, \log_2 n$ **do**
2: Peggy chooses a random $v \in Z_n^*$ and sends $y = v^2 \bmod n$ to Victor;
3: Victor chooses a random bit i and sends it to Peggy;
4: Peggy sends $z = u^i v \bmod n$, where u is a square root of x; $\{u^2 \equiv x \bmod n.\}$
5: Victor checks if $z^2 \equiv x^i y \bmod n$;
6: **end for**
7: Victor accepts x if Line 5 is confirmed every time;

A Useful Corollary

Corollary 76 *Let $n = pq$ be a product of two distinct primes. (1) If x and y are both quadratic residues modulo n, then $xy \in Z_n^*$ is a quadratic residue modulo n. (2) If x is a quadratic residue modulo n and y is a quadratic nonresidue modulo n, then $xy \in Z_n^*$ is a quadratic nonresidue modulo n.*

- Suppose x and y are both quadratic residues modulo n.
- Let $x \equiv a^2 \bmod n$ and $y \equiv b^2 \bmod n$.
- Now xy is a quadratic residue as $xy \equiv (ab)^2 \bmod n$.

The Proof (concluded)

- Suppose x is a quadratic residue modulo n and y is a quadratic nonresidue modulo n.
- By Lemma 75 (p. 586), $(x \,|\, p) = (x \,|\, q) = 1$ but, say, $(y \,|\, p) = -1$.
- Now xy is a quadratic nonresidue as $(xy \,|\, p) = -1$, again by Lemma 75 (p. 586).

Analysis

- Suppose x is a quadratic nonresidue.
  - Peggy can answer only one of the two possible challenges.
    - If a is a quadratic residue, then xa is a quadratic nonresidue by Corollary 76 (p. 614).
    - So $x^i y$ can be a quadratic residue (see Line 5) only when $i = 0$.
  - So Peggy will be caught in any given round with probability one half.

Analysis (continued)

- Suppose x is a quadratic residue.
  - Peggy can answer all challenges.
  - So Victor will accept x.
- How about the claim of zero knowledge?
- The transcript between Peggy and Victor when x is a quadratic residue can be generated without Peggy!
- Here is how.

Analysis (continued)

- Suppose x is a quadratic residue.[a]
- In each round of interaction with Peggy, the transcript is a triplet (y, i, z).
- We present an efficient Bob that generates (y, i, z) with the same probability *without* accessing Peggy.

[a]There is no zero-knowledge requirement when $x \notin L$.

Analysis (concluded)

1: Bob chooses a random $z \in Z_n^*$;

2: Bob chooses a random bit i;

3: Bob calculates $y = z^2 x^{-i} \bmod n$;

4: Bob writes (y, i, z) into the transcript;

Comments

- Assume x is a quadratic residue.
- In both cases, for (y, i, z), y is a random quadratic residue, i is a random bit, and z is a random number.
- Bob cheats because (y, i, z) is *not* generated in the same order as in the original transcript.
  - Bob picks Peggy's answer z first.
  - Bob then picks Victor's challenge i.
  - Bob finally patches the transcript.

Comments (concluded)

- So it is not the transcript that convinces Victor, but that conversation with Peggy is held "on line."
- The same holds even if the transcript was generated by a cheating Victor's interaction with (honest) Peggy.
- But we skip the details.

Does the Following Work, Too?[a]

```
1: for m = 1, 2, ..., log_2 n do
2:     Peggy chooses a random v ∈ Z_n^* and sends
       y = v^2 mod n to Victor;
3:     Peggy sends z = uv mod n, where u is a square root of
       x; {u^2 ≡ x mod n.}
4:     Victor checks if z^2 ≡ xy mod n;
5: end for
6: Victor accepts x if Line 4 is confirmed every time;
```

[a]Contributed by Mr. Chih-Duo Hong (R95922079) on December 13, 2006. It is like always choosing $i = 1$ in the original protocol.

Does the Following Work, Too?[a] (concluded)

- Suppose x is a quadratic nonresidue.
- But Peggy can mislead Victor into accepting x as a quadratic residue.
- She simply sends $y = x$ and $z = x$ to Victor.
- This pair will satisfy $z^2 \equiv xy \bmod n$ by construction.
- The protocol is hence not even an IP protocol!

[a]Contributed by Mr. Chin-Luei Chang (D95922007) on June 16, 2008.

Zero-Knowledge Proof of 3 Colorability[a]

for $i = 1, 2, \ldots, |E|^2$ **do**
    Peggy chooses a random permutation π of the 3-coloring ϕ;
    Peggy samples encryption schemes randomly, commits[b] them, and sends $\pi(\phi(1)), \pi(\phi(2)), \ldots, \pi(\phi(|V|))$ *encrypted* to Victor;
    Victor chooses at random an edge $e \in E$ and sends it to Peggy for the coloring of the endpoints of e;
    if $e = (u, v) \in E$ **then**
        Peggy reveals the coloring of u and v and "proves" that they correspond to their encryptions;
    else
        Peggy stops;
    end if

---

[a] Goldreich, Micali, and Wigderson (1986).

[b] Contributed by Mr. Ren-Shuo Liu (`D98922016`) on December 22, 2009.

10: **if** the "proof" provided in Line 6 is not valid **then**
11: Victor rejects and stops;
12: **end if**
13: **if** $\pi(\phi(u)) = \pi(\phi(v))$ or $\pi(\phi(u)), \pi(\phi(v)) \not\in \{1, 2, 3\}$ **then**
14: Victor rejects and stops;
15: **end if**
16: **end for**
17: Victor accepts;

Analysis

- If the graph is 3-colorable and both Peggy and Victor follow the protocol, then Victor always accepts.
- Suppose the graph is not 3-colorable and Victor follows the protocol.
- Let e be an edge that is not colored legally, which Victor will pick with probability $1/m$, where $m = |\, E \,|$.
- Then however Peggy plays, Victor will accept with probability $\leq 1 - 1/m$ per round.
- So Victor will accept with probability $\leq (1 - 1/m)^{m^2} \leq e^{-m}$.
- Thus the protocol is valid.

Analysis (concluded)

- This protocol yields no knowledge to Victor as all he gets is a bunch of random pairs.
- The proof that the protocol is zero-knowledge to *any* verifier is intricate.

Comments

- Each $\pi(\phi(i))$ is encrypted by a different cryptosystem in Line 3.[a]
  - Otherwise, all the colors will be revealed in Step 6.
- Each edge e must be picked randomly.[b]
  - Otherwise, Peggy will know Victor's game plan and plot accordingly.

---

[a] Contributed by Ms. Yui-Huei Chang (R96922060) on May 22, 2008

[b] Contributed by Mr. Chang-Rong Hung (R96922028) on May 22, 2008