Theory of Computation

Solutions to Homework 1

Problem 1. Two disjoint languages $\mathcal{L}_1$ and $\mathcal{L}_2$ are called recursively separable if there exists a recursive language $\mathcal{R}$ such that $\mathcal{L}_1 \cap \mathcal{R} = \emptyset$ and $\mathcal{L}_2 \subseteq \mathcal{R}$. Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are recursively separable languages. Show that if both $\mathcal{L}_1$ and $\bar{\mathcal{L}}_1 \cup \mathcal{L}_2$ are recursively enumerable, then $\mathcal{L}_1$ is recursive.

Proof. Without loss of generality, assume $\mathcal{L}_1$ and $\mathcal{L}_2$ are recursively separable with $\mathcal{L}_2 \subseteq \mathcal{R}$. Obviously, $\mathcal{L}_2 \subseteq \bar{\mathcal{L}}_1$. Let TM $\mathcal{M}$ accept $\bar{\mathcal{L}}_1 \cup \mathcal{L}_2$. Because $\bar{\mathcal{L}}_1 \cup \mathcal{L}_2 = \bar{\mathcal{L}}_1$, $\bar{\mathcal{L}}_1$ is also accepted by $\mathcal{M}$, thus recursively enumerable. Using the idea in Lemma 10 on p. 131, it implies that $\mathcal{L}_1$ is recursive.

Problem 2. Prove that the subsets of distinct primes form an uncountable set.

Proof. Denote the nth prime as p_n. It is easy to show that there is a bijection between $\mathbb{N}$ and primes $f : \{1,2,3,4,\dots,n,\dots\} \to \{2,3,5,7,\dots,p_n,\dots\}$. Therefore, primes are countable. Thus the problem is equivalent to asking whether a function g exists such that

$$g : \mathbb{N} \to 2^{\mathbb{N}}$$

is a bijection. Cantor's theory says no such g exists. Hence the subsets of distinct primes do not form a countable set.