# Theory of Computation

Final Examination on January 11, 2011
Fall Semester, 2010

**Problem 1 (25 points)** Let $A, B$ be finite nonempty sets, $f : A \times B \to \{0,1\}$ and $\sum_{y\in B} f(x,y) < |\,B\,|/|\,A\,|$ for all $x \in A$. Prove the existence of a $y^* \in B$ with $\sum_{x\in A} f(x,y^*) = 0$. You may want to use the fact

$$\sum_{x\in A}\sum_{y\in B} f(x,y) = \sum_{y\in B}\sum_{x\in A} f(x,y).$$

**Ans:** As $\sum_{y\in B} f(x,y) < |\,B\,|/|\,A\,|$ for $x \in A$,

$$\sum_{x\in A}\sum_{y\in B} f(x,y) < \sum_{x\in A} \frac{|\,B\,|}{|\,A\,|} = |\,B\,|. \tag{1}$$

Suppose for contradiction that

$$\sum_{x\in A} f(x,y) \geq 1$$

for all $y \in B$. Then

$$\sum_{y\in B}\sum_{x\in A} f(x,y) \geq \sum_{y\in B} 1 = |\,B\,|,$$

contradicting inequality (1). ■

**Problem 2 (25 points)** Does IP contain all languages that have uniformly polynomial circuits?

**Ans:** Yes. P equals the class of languages with uniformly polynomial circuits. Furthermore, any language in P can be decided by an interactive proof system where the verifier simply decides the language itself and ignores the prover's messages. So P $\subseteq$ IP. ■

**Problem 3 (25 points)** Show that if NP $\neq$ coNP, then P $\neq$ NP.

**Ans:** P is closed under complementation. If P = NP, then NP is closed under complementation. In other words, NP = coNP. This is the contrapositive of the assumption. ■

**Problem 4 (25 points)** FP is the set of polynomial-time computable functions. GCD, LCM, matrix-matrix multiplication, etc. are in FP. Let #SAT stand for the problem of calculating the number of satisfying truth assignments to a boolean formula. Show that if #SAT $\in$ FP, then P = NP.

**Ans:** Given boolean formula $\phi$, calculate its number of satisfying truth assignments, $k$, in polynomial time. Declare "$\phi \in$ SAT" if and only if $k \geq 1$. ■