Theory of Computation

Solutions to Homework 3

Problem 1. Show that if a coNP-complete problem is in NP, then NP = coNP.

Proof. Suppose $L' \in$ NP is coNP-complete. Let NTM M decide L':
1) For $x \in L'$, $M(x) =$ "yes" for some computation path.
2) For $x \notin L'$, $M(x) =$ "no" for all computation paths.
Note that M shows $L' \in$ NP. Then every $L \in$ coNP is reducible to L'. Let R be a reduction from L to L' such that
1) for $x \in L$, $R(x) \in L'$.
2) for $x \notin L$, $R(x) \notin L'$.
$L \in$ NP as it is decided by the nondeterministic polynomial-time algorithm $M(R(x))$, because
1) For $x \in L$, $M(R(x)) =$ "yes" for some computation path.
2) For $x \notin L$, $M(R(x)) =$ "no" for all computation paths.
So coNP $\subseteq$ NP. The other direction NP $\subseteq$ coNP is symmetric. So, NP = coNP. □

Problem 2. It is known that 3-coloring is NP-complete. Show that 4-coloring is NP-complete. (You do not need to show that it is in NP.)

Proof. We give a polynomial-time reduction from 3-COLOR to 4-COLOR. The reduction maps a graph G into a new graph G' such that $G \in$ 3-COLOR if and only if $G' \in$ 4-COLOR. We do so by setting G' to G, and then adding a new node y and connecting y to each node in G'. If G is 3-colorable, then G' can be 4-colored exactly as G with y being the only node colored with the additional color. Similarly, if G' is 4-colorable, then we know that node y must be the only node of its color 4 this is because it is connected to every other node in G'. Thus, we know that G must be 3-colorable. This reduction takes linear time to add a single node and G edges. □