Decidability and Recursive Languages

- Let $L \subseteq (\Sigma - \{\sqcup\})^*$ be a **language**, i.e., a set of strings of symbols with a *finite* length.
  - For example, $\{0, 01, 10, 210, 1010, \ldots\}$.
- Let M be a TM such that for any string x:
  - If $x \in L$, then $M(x) =$ "yes."
  - If $x \notin L$, then $M(x) =$ "no."
- We say M **decides** L.
- If L is decided by some TM, then L is **recursive**.

Recursive Languages: Examples

- The set of palindromes over any alphabet is recursive.
- The set of prime numbers $\{2, 3, 5, 7, 11, 13, 17, \ldots\}$ is recursive.
- The set of C programs that do not contain a `while`, a `for`, or a `goto` is recursive.
- The set of C programs that do not contain an infinite loop is *not* recursive (to be proved later).

Acceptability and Recursively Enumerable Languages

- Let $L \subseteq (\Sigma - \{\sqcup\})^*$ be a language.
- Let M be a TM such that for any string x:
  - If $x \in L$, then $M(x) =$ "yes."
  - If $x \notin L$, then $M(x) = \nearrow$.
- We say M **accepts** L.

Acceptability and Recursively Enumerable Languages (concluded)

- If L is accepted by some TM, then L is called a **recursively enumerable language**.[a]
  - A recursively enumerable language can be generated by a TM, thus the name.
  - That is, there is an algorithm such that for every $x \in L$, it will be printed out eventually.
  - This algorithm may never terminate.

---

[a] Post (1944).

Emil Post (1897–1954)

Recursive and Recursively Enumerable Languages

Proposition 1 *If L is recursive, then it is recursively enumerable.*

- Let TM M decide L.
- We need to design a TM that accepts L.
- We next modify M's program to obtain M' that accepts L.
- M' is identical to M except that when M is about to halt with a "no" state, M' goes into an infinite loop.
- M' accepts L.

Recursively Enumerable Languages: Examples

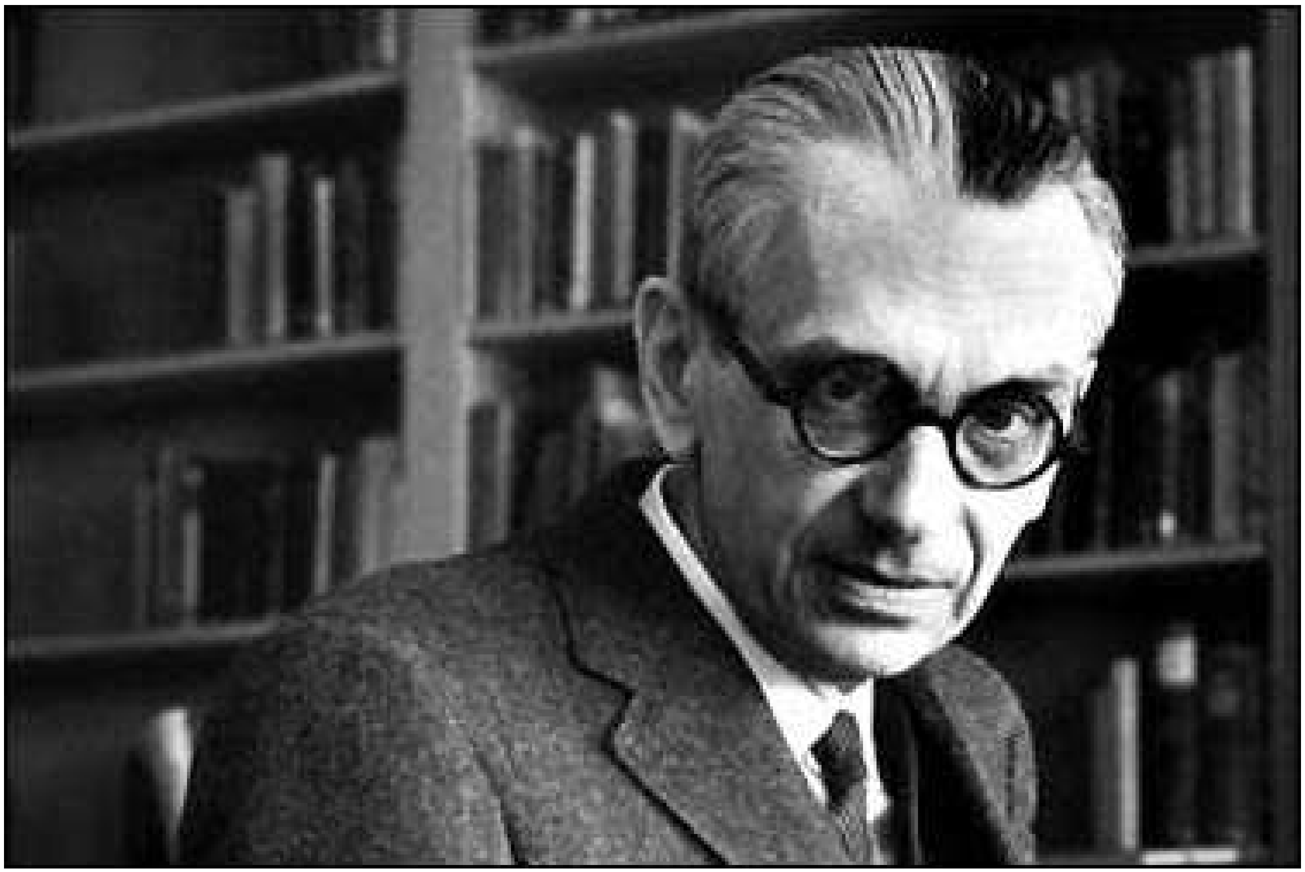
- The set of C program-input pairs that do run into an infinite loop is recursively enumerable.
  - Just run it in a simulator environment.
- The set of C programs that contain an infinite loop is *not* recursively enumerable (to be proved later).
- The set of valid statements of an axiomatic system is recursively enumerable.
  - Try all possible proofs systematically.

Turing-Computable Functions

- Let $f : (\Sigma - \{\bigsqcup\})^* \to \Sigma^*$.
  - Optimization problems, root finding problems, etc.
- Let M be a TM with alphabet Σ.
- M **computes** f if for any string $x \in (\Sigma - \{\bigsqcup\})^*$, $M(x) = f(x)$.
- We call f a **recursive function**[a] if such an M exists.

[a] Kurt Gödel (1931).

Kurt Gödel (1906–1978)



Church's Thesis or the Church-Turing Thesis

- What is computable is Turing-computable; TMs are algorithms.[a]
- Many other computation models have been proposed.
  - Recursive function (Gödel), λ calculus (Church), formal language (Post), assembly language-like RAM (Shepherdson & Sturgis), boolean circuits (Shannon), extensions of the Turing machine (more strings, two-dimensional strings, and so on), etc.
- All have been proved to be equivalent.

---

[a]Kleene (1953).

Church's Thesis or the Church-Turing Thesis (concluded)

- No "intuitively computable" problems have been shown not to be Turing-computable yet.
- The thesis is[a]

> a profound claim about the physical laws of our universe, i.e.: any physical system that purports to be a computer is not capable of any computational task that a Turing machine is incapable of.

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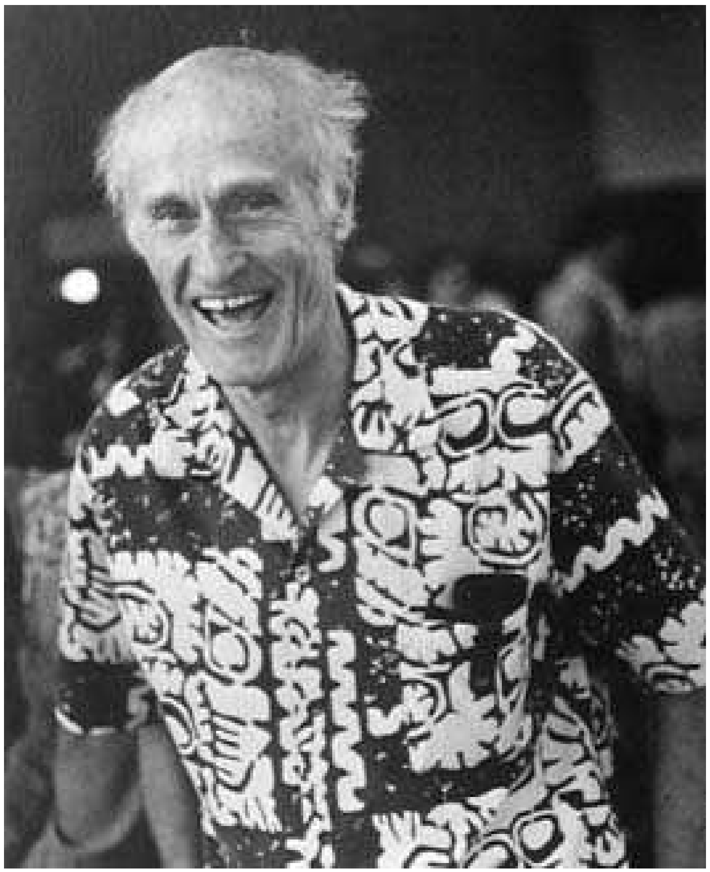
[a]Smith (1998).

Alonso Church (1903–1995)

Stephen Kleene (1909–1994)



Extended Church's Thesis[a]

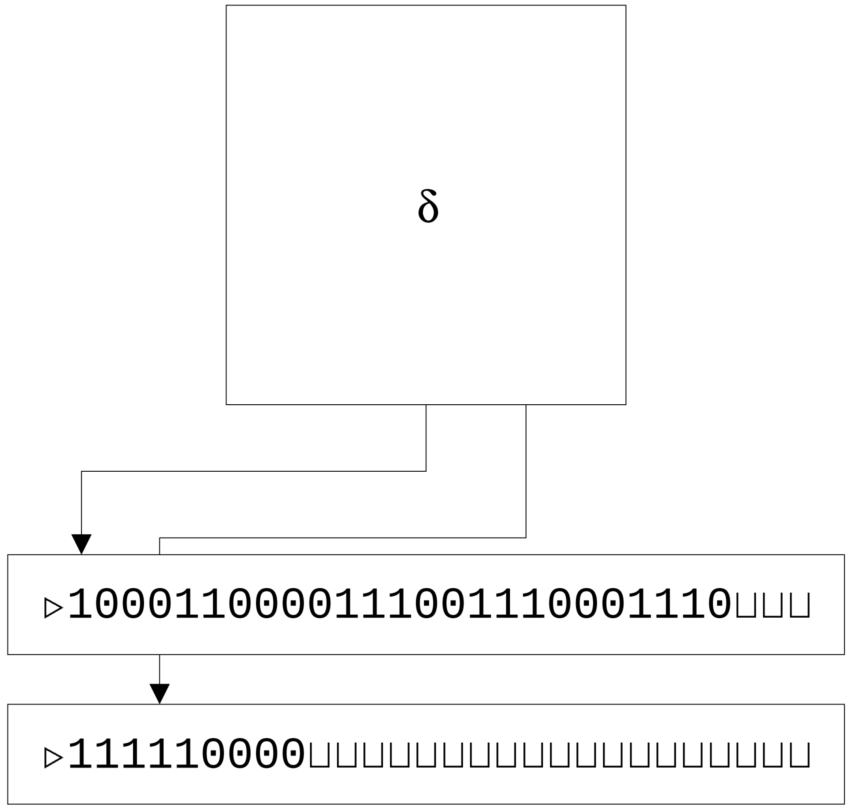
- All "reasonably succinct encodings" of problems are *polynomially related.*
  - Representations of a graph as an adjacency matrix and as a linked list are both succinct.
  - The *unary* representation of numbers is not succinct.
  - The *binary* representation of numbers is succinct.
    - 1001 vs. 111111111.
- All numbers for TMs will be binary from now on.

[a]Some call it "polynomial Church's thesis," which Lószló Lovász attributed to Leonid Levin.

Turing Machines with Multiple Strings

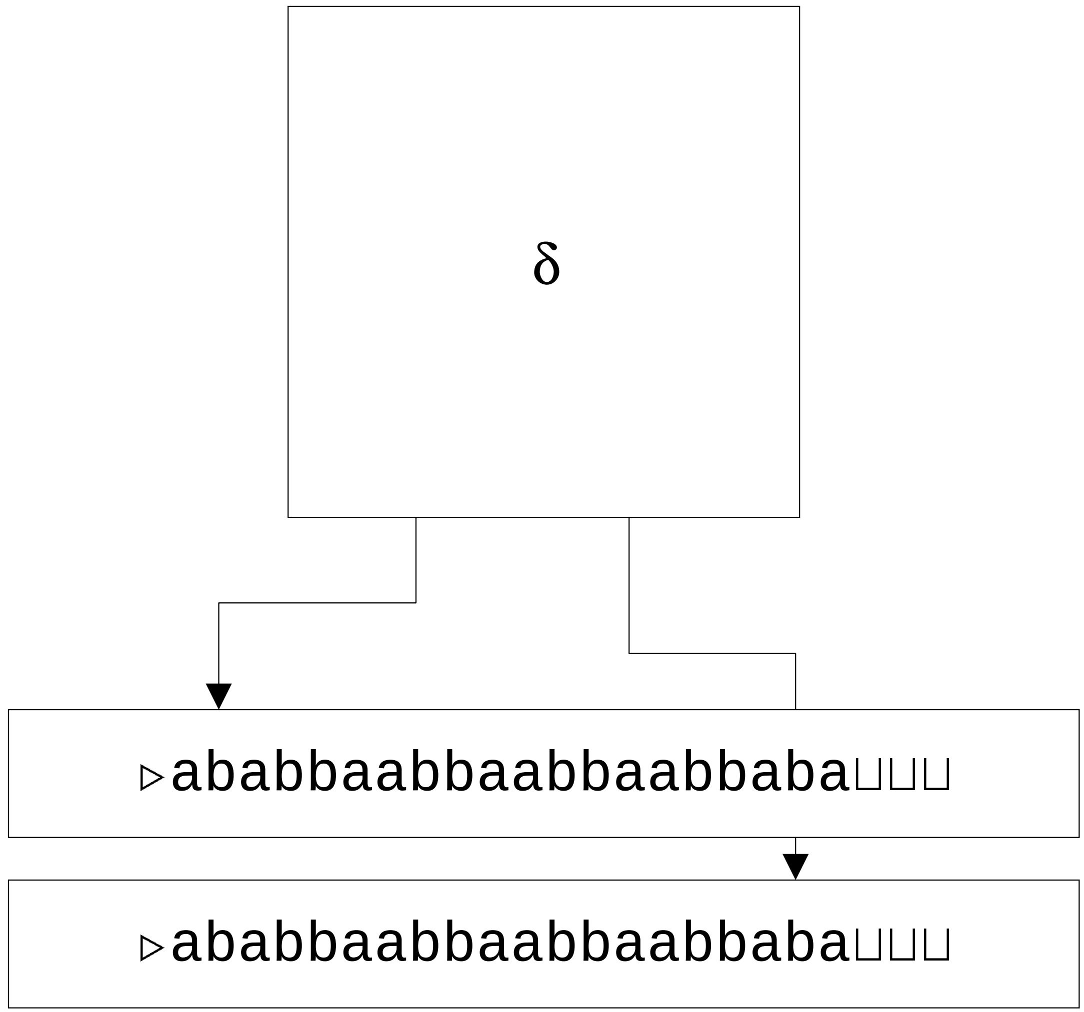
- A k-string Turing machine (TM) is a quadruple $M = (K, \Sigma, \delta, s)$.
- K, Σ, s are as before.
- $\delta : K \times \Sigma^k \to (K \cup \{h, \text{“yes”}, \text{“no”}\}) \times (\Sigma \times \{\leftarrow, \rightarrow, -\})^k$.
- All strings start with a $\rhd$.
- The first string contains the input.
- Decidability and acceptability are the same as before.
- When TMs compute functions, the output is on the last (kth) string.

A 2-String TM



PALINDROME Revisited

- A 2-string TM can decide PALINDROME in $O(n)$ steps.
  - It copies the input to the second string.
  - The cursor of the first string is positioned at the first symbol of the input.
  - The cursor of the second string is positioned at the last symbol of the input.
  - The two cursors are then moved in opposite directions until the ends are reached.
  - The machine accepts if and only if the symbols under the two cursors are identical at all steps.


δ
▷ababbaabbaabbaabbaba⊔⊔⊔
▷ababbaabbaabbaabbaba⊔⊔⊔

Configurations and Yielding

- The concept of configuration and yielding is the same as before except that a configuration is a $(2k+1)$-tuple
$$(q, w_1, u_1, w_2, u_2, \ldots, w_k, u_k).$$
  - $w_i u_i$ is the ith string.
  - The ith cursor is reading the last symbol of w_i.
  - Recall that $\rhd$ is each w_i's first symbol.
- The k-string TM's initial configuration is
$$(s, \overbrace{\underbrace{\rhd, x}_{1}, \underbrace{\rhd, \epsilon}_{2}, \underbrace{\rhd, \epsilon}_{3}, \ldots, \underbrace{\rhd, \epsilon}_{k}}^{2k}).$$

Time Complexity

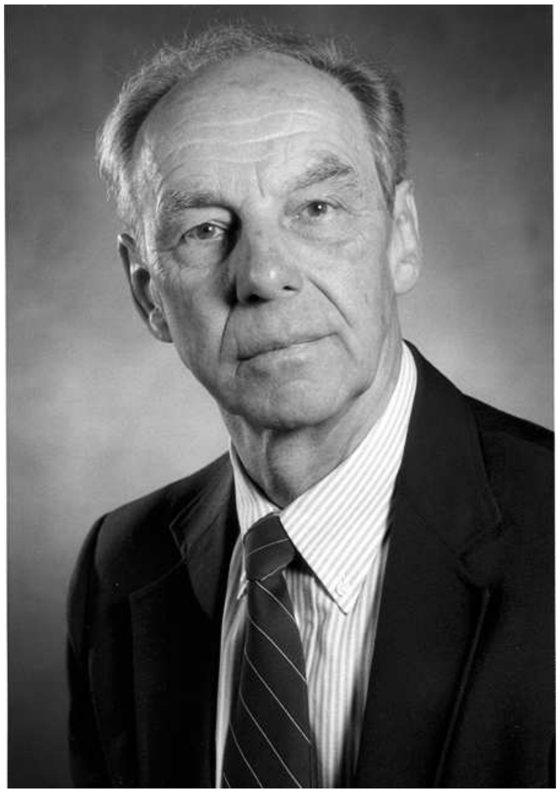
- The multistring TM is the basis of our notion of the time expended by TM computations.
- If a k-string TM M halts after t steps on input x, then the **time required by M on input x** is t.
- If $M(x) = \nearrow$, then the time required by M on x is ∞.
- Machine M **operates within time** $f(n)$ for $f : \mathbb{N} \to \mathbb{N}$ if for any input string x, the time required by M on x is at most $f(|\,x\,|)$.
  - $|\,x\,|$ is the length of string x.
- Function $f(n)$ is a **time bound** for M.

Time Complexity Classes[a]

- Suppose language $L \subseteq (\Sigma - \{\sqcup\})^*$ is decided by a multistring TM operating in time $f(n)$.
- We say $L \in \mathrm{TIME}(f(n))$.
- $\mathrm{TIME}(f(n))$ is the set of languages decided by TMs with multiple strings operating within time bound $f(n)$.
- $\mathrm{TIME}(f(n))$ is a **complexity class**.
  - PALINDROME is in $\mathrm{TIME}(f(n))$, where $f(n) = O(n)$.

[a]Hartmanis and Stearns (1965); Hartmanis, Lewis, and Stearns (1965).

Juris Hartmanis[a] (1928–)



[a]Turing Award (1993).

Richard Edwin Stearns[a] (1936–)

[a]Turing Award (1993).

The Simulation Technique

Theorem 2 *Given any k-string M operating within time $f(n)$, there exists a (single-string) M' operating within time $O(f(n)^2)$ such that $M(x) = M'(x)$ for any input x.*

- The single string of M' implements the k strings of M.
- Represent configuration $(q, w_1, u_1, w_2, u_2, \ldots, w_k, u_k)$ of M by configuration

$$(q, \triangleright w_1' u_1 \lhd w_2' u_2 \lhd \cdots \lhd w_k' u_k \lhd \lhd)$$

  of M'.
  - $\lhd$ is a special delimiter.
  - w_i' is w_i with the first[a] and last symbols "primed."

[a]The first symbol is always $\triangleright$.

The Proof (continued)

- The "priming" of the last symbol of w_i ensures that M' knows which symbol is under the cursor for each simulated string.[a]
- We use the primed version of the first symbol of w_i (so $\rhd$ becomes $\rhd'$).
- Recall the requirement on p. 20 that $\delta(q, \rhd) = (p, \rhd, \rightarrow)$ so that the cursor is not allowed to move to the left of $\rhd$.
- So the single cursor of M' can move *between* the simulated strings of M.[b]

---

[a] Added because of comments made by Mr. Che-Wei Chang (R95922093) on September 27, 2006.

[b] Thanks to a lively discussion on September 22, 2009.

The Proof (continued)

- The initial configuration of M' is

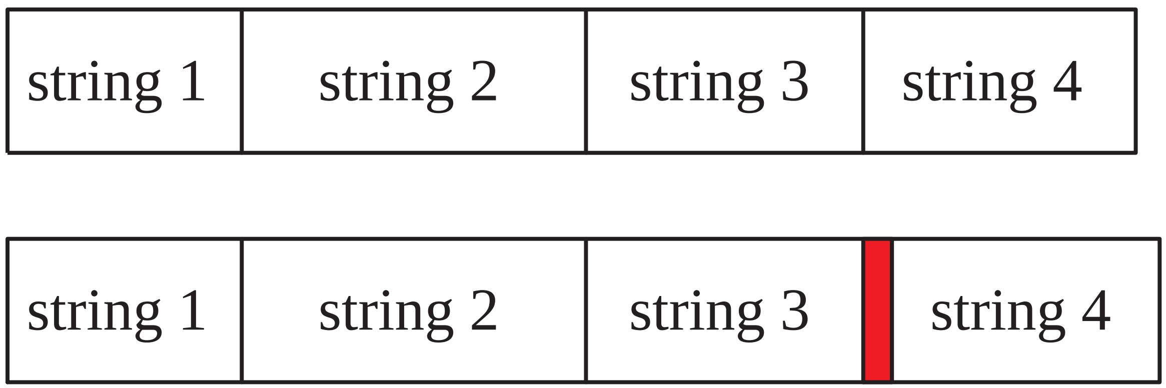
$$(s, \rhd\, \rhd'\, x \lhd \overbrace{\rhd' \lhd \cdots \rhd' \lhd}^{k-1 \text{ pairs}} \lhd).$$

- We simulate each move of M thus:
  1. M' scans the string to pick up the k symbols under the cursors.
     - The states of M' must be enlarged to include $K \times \Sigma^k$ to remember them.
     - The transition functions of M' must also reflect it.
  2. M' then changes the string to reflect the overwriting of symbols and cursor movements of M.

The Proof (continued)

- It is possible that some strings of M need to be lengthened (see next page).
  - The linear-time algorithm on p. 34 can be used for each such string.
- The simulation continues until M halts.
- M' then erases all strings of M except the last one.
- Since M halts within time $f(|x|)$, none of its strings ever becomes longer than $f(|x|)$.[a]
- The length of the string of M' at any time is $O(kf(|x|))$.

[a] We tacitly assume $f(n) \geq n$.


string 1
string 2
string 3
string 4
string 1
string 2
string 3
string 4

The Proof (concluded)

- Simulating each step of M takes, *per string of* M, $O(kf(|x|))$ steps.
  - $O(f(|x|))$ steps to collect information.
  - $O(kf(|x|))$ steps to write and, if needed, to lengthen the string.
- M' takes $O(k^2 f(|x|))$ steps to simulate each step of M because there are k strings.
- As there are $f(|x|)$ steps of M to simulate, M' operates within time $O(k^2 f(|x|)^2)$.

Linear Speedup[a]

Theorem 3 *Let* $L \in \mathit{TIME}(f(n))$*. Then for any* $\epsilon > 0$*,* $L \in \mathit{TIME}(f'(n))$*, where* $f'(n) = \epsilon f(n) + n + 2$*.*

[a] Hartmanis and Stearns (1965).

Implications of the Speedup Theorem

- State size can be traded for speed.
  - $m^k \cdot |\Sigma|^{3mk}$-fold increase to gain a speedup of $O(m)$.
- If $f(n) = cn$ with $c > 1$, then c can be made arbitrarily close to 1.
- If $f(n)$ is superlinear, say $f(n) = 14n^2 + 31n$, then the constant in the leading term (14 in this example) can be made arbitrarily small.
  - *Arbitrary* linear speedup can be achieved.[a]
  - This justifies the big-O notation for the analysis of algorithms.

---

[a]Can you apply the theorem multiple times to achieve superlinear speedup? Thanks to a question by a student on September 21, 2010.

P

- By the linear speedup theorem, any polynomial time bound can be represented by its leading term n^k for some $k \geq 1$.
- If L is a polynomially decidable language, it is in $\text{TIME}(n^k)$ for some $k \in \mathbb{N}$.
  - Clearly, $\text{TIME}(n^k) \subseteq \text{TIME}(n^{k+1})$.
- The union of all polynomially decidable languages is denoted by P:

$$\text{P} = \bigcup_{k>0} \text{TIME}(n^k).$$

- P contains problems that can be efficiently solved.

Space Complexity

- Consider a k-string TM M with input x.
- Assume non-$\sqcup$ is never written over by $\sqcup$.[a]
  - The purpose is not to artificially downplay the space requirement.
- If M halts in configuration $(H, w_1, u_1, w_2, u_2, \ldots, w_k, u_k)$, then the **space required by M on input x** is $\sum_{i=1}^{k} |w_i u_i|$.

---

[a]Corrected by Ms. Chuan-Ju Wang (`R95922018`) on September 27, 2006.

Space Complexity (continued)

- Suppose we do not charge the space used only for input and output.
- Let $k > 2$ be an integer.
- A **k-string Turing machine with input and output** is a k-string TM that satisfies the following conditions.
  - The input string is *read-only*.
  - The last string, the output string, is *write-only*.
  - So the cursor never moves to the left.
  - The cursor of the input string does not wander off into the $\sqcup$s.

Space Complexity (concluded)

- If M is a TM with input and output, then the space required by M on input x is $\sum_{i=2}^{k-1} |w_i u_i|$.
- Machine M **operates within space bound** $f(n)$ for $f : \mathbb{N} \to \mathbb{N}$ if for any input x, the space required by M on x is at most $f(|x|)$.

Space Complexity Classes

- Let L be a language.
- Then
$$L \in \text{SPACE}(f(n))$$
if there is a TM with input and output that decides L and operates within space bound $f(n)$.
- SPACE($f(n)$) is a set of languages.
  - PALINDROME $\in$ SPACE($\log n$): Keep 3 counters.
- As in the linear speedup theorem (Theorem 3), constant coefficients do not matter.

Nondeterminism[a]

- A **nondeterministic Turing machine** (**NTM**) is a quadruple $N = (K, \Sigma, \Delta, s)$.
- K, Σ, s are as before.
- $\Delta \subseteq K \times \Sigma \times (K \cup \{h, \text{"yes"}, \text{"no"}\}) \times \Sigma \times \{\leftarrow, \rightarrow, -\}$ is a relation, not a function.[b]
  - For each state-symbol combination, there may be multiple valid next steps—or none at all.
  - Multiple instructions may be applicable.

---

[a]Rabin and Scott (1959).

[b]Corrected by Mr. Jung-Ying Chen (`D95723006`) on September 23, 2008.

Nondeterminism (concluded)

- As before, a program contains lines of codes:

$$\begin{aligned} (q_1, \sigma_1, p_1, \rho_1, D_1) &\in \Delta, \\ (q_2, \sigma_2, p_2, \rho_2, D_2) &\in \Delta, \\ &\vdots \\ (q_n, \sigma_n, p_n, \rho_n, D_n) &\in \Delta. \end{aligned}$$

  - In the deterministic case (p. 21), we wrote

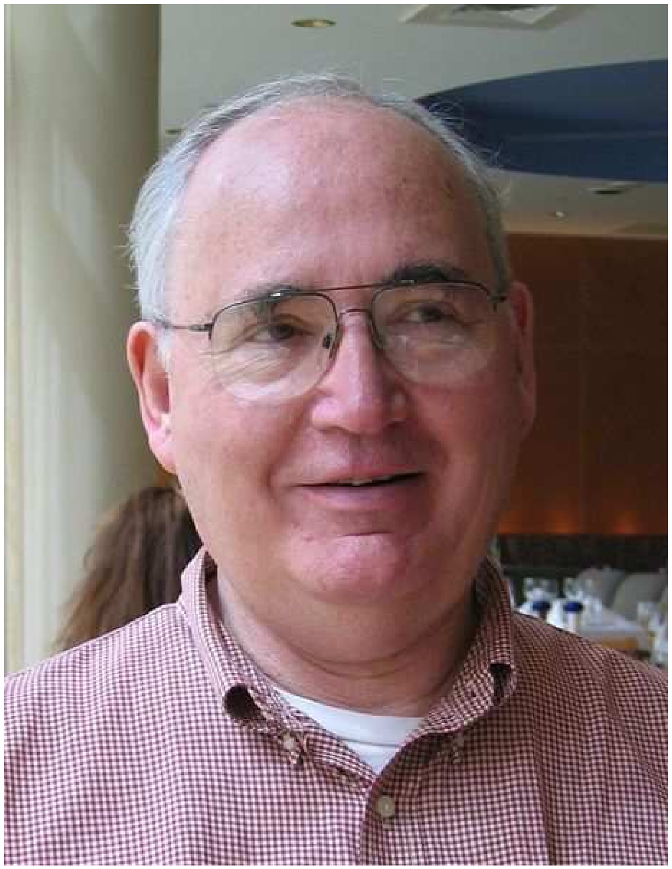
$$\delta(q_i, \sigma_i) = (p_i, \rho_i, D_i).$$

- A configuration yields another configuration in one step if there *exists* a rule in Δ that makes this happen.

Michael O. Rabin[a] (1931–)

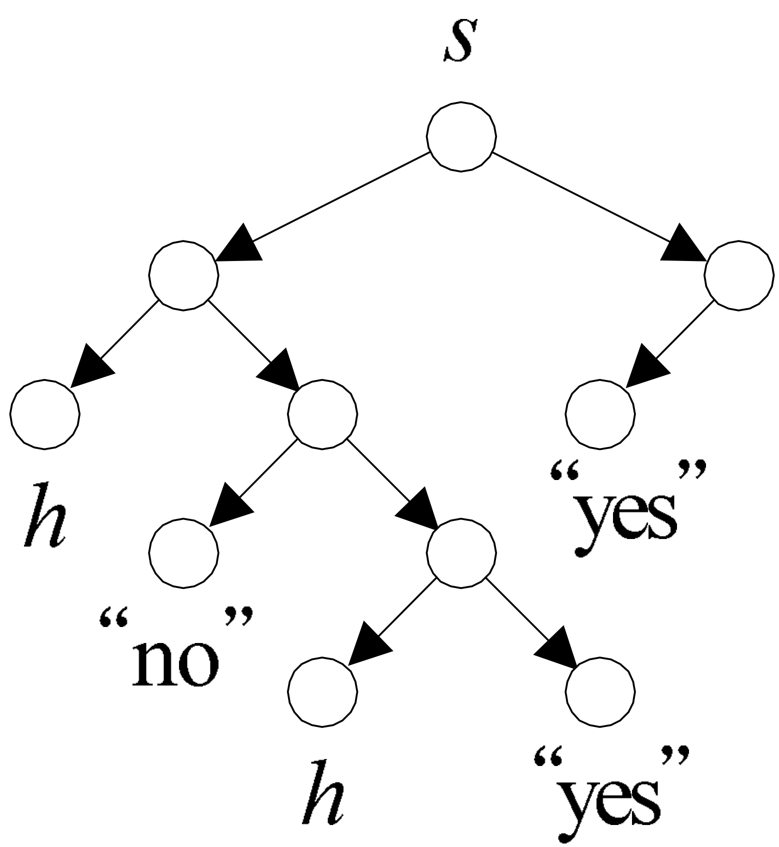
[a]Turing Award (1976).

Dana Stewart Scott[a] (1932–)



[a]Turing Award (1976).

Computation Tree and Computation Path



Decidability under Nondeterminism

- Let L be a language and N be an NTM.
- N **decides** L if for any $x \in \Sigma^*$, $x \in L$ if and only if there is a sequence of valid configurations that ends in "yes."
  - It is not required that the NTM halts in all computation paths.[a]
  - If $x \notin L$, no nondeterministic choices should lead to a "yes" state.
- What is key is the algorithm's overall behavior not whether it gives a correct answer for each particular run.
- Determinism is a special case of nondeterminism.

[a] So "accepts" may be a more proper term.

An Example

- Let L be the set of logical conclusions of a set of axioms.
  - Predicates not in L may be false under the axioms.
  - They may also be independent of the axioms.
    * That is, they can be assumed true or false without contradicting the axioms.

An Example (concluded)

- Let ϕ be a predicate whose validity we would like to prove.
- Consider the nondeterministic algorithm:

```
b := true;
while the input predicate φ ≠ b do
    Generate a logical conclusion of b by applying one
    of the axioms; {Nondeterministic choice.}
    Assign this conclusion to b;
end while
"yes";
```

- This algorithm decides L.

