Theory of Computation

Homework 3
Due: 9:10, 2009/11/24

Problem 1. Prove that if $\mathrm{coNP} \neq \mathrm{NP}$, then $\mathrm{P} \neq \mathrm{NP}$.

Proof. If $\mathrm{P} = \mathrm{NP}$, then $\mathrm{coP} = \mathrm{coNP}$; hence

$$\mathrm{coNP} = \mathrm{coP} = \mathrm{P} = \mathrm{NP}.$$

So $\mathrm{coNP} \neq \mathrm{NP}$ implies $\mathrm{P} \neq \mathrm{NP}$. ■

Problem 2. It is known that the 3-COLORING problem is NP-complete. Use this fact to prove that for any given $\mathrm{k} > 3$, it is NP-hard to ask if a graph can be colored by k or fewer colors such that no adjacent nodes have the same color.

Proof. We show a reduction from 3-COLORING to k-COLORING, i.e., the problem of asking if a graph can be colored by k or fewer colors such that no adjacent nodes have the same color. Given a graph $\mathrm{G}(\mathrm{V}, \mathrm{E})$, the reduction outputs a graph $\mathrm{G}'(\mathrm{V}', \mathrm{E}')$ by adding $\mathrm{k} - 3$ new nodes and all edges with any of them as an endpoint. That is, $\mathrm{V}' = \mathrm{V} \cup \{\mathrm{x}_1, \dots, \mathrm{x}_{\mathrm{k}-3}\}$ and $\mathrm{E}' = \mathrm{E} \cup \{(x_i, v) | v \in V', 1 \leq i \leq k - 3\}$, where $\mathrm{x}_\mathrm{i} \notin \mathrm{V}$ for $1 \leq \mathrm{i} \leq \mathrm{k} - 3$. If $\mathrm{G} \in \text{3-COLORING}$, then $\mathrm{G}' \in \text{k-COLORING}$ because 3 or fewer colors for the nodes in V and additional $\mathrm{k} - 3$ colors for those in $\{\mathrm{x}_1, \dots, \mathrm{x}_{\mathrm{k}-3}\}$ are needed so that no adjacent nodes have the same color. Conversely, consider a coloring of G' with k or fewer colors such that no adjacent nodes have the same color. In such a coloring, $\mathrm{x}_1, \dots, \mathrm{x}_{\mathrm{k}-3}$ use up exactly $\mathrm{k} - 3$ colors, leaving at most 3 colors for the nodes in V. ■