Theory of Computation

Mid-Term Examination on November 10, 2009

Problem 1 (25 points)**.** Show that if NP $\subseteq$ NSPACE(n^2), then NP $\neq$ PSPACE.

Proof. By Savitch's theorem, NSPACE(n^2) $\subseteq$ SPACE(n^4). By the space-hierarchy theorem, SPACE(n^4) $\subsetneq$ PSPACE. Hence NP $\subseteq$ NSPACE(n^2) implies

$$\text{NP} \subseteq \text{NSPACE}(n^2) \subseteq \text{SPACE}(n^4) \subsetneq \text{PSPACE}.$$

□

Problem 2 (25 points)**.** Let the MIX HAMILTONIAN PATH problem ask whether, given two undirected graphs, exactly one of them has a Hamiltonian path. Prove or disprove that MIX HAMILTONIAN PATH is NP-hard.

Proof. We present a logarithmic-space reduction from HAMILTONIAN PATH to MIX HAMILTONIAN PATH, thus establishing the NP-hardness of MIX HAMILTONIAN PATH. Let G_0 be a fixed undirected graph without a Hamiltonian path. Given an undirected graph G, the reduction outputs G and G_0. Clearly, G has a Hamiltonian path if and only if exactly one of G and G_0 does. □

Problem 3 (25 points)**.** It is known that EXP-hard languages exist. Can every NP-complete language be reduced to an EXP-hard language? Briefly justify your answer.

Proof. Clearly, NP $\subseteq$ EXP. By definition, all languages in EXP, including the NP-complete ones, can be reduced to an EXP-hard language. □

Problem 4 (25 points)**.** Show that if both L and $\bar{L}$ are recursively enumerable languages, then L is recursive.

Proof. Suppose that L and $\bar{L}$ are accepted by Turing machines M and $\bar{M}$, respectively. Then L is decided by Turing machine M', defined as follows. On input x, M' simulates on two different strings both M and $\bar{M}$ in an interleaved fashion. That is, it simulates a step of M on one string, then a step of $\bar{M}$ on the other, then again another step of M on the first, and so on. Since M accepts L, $\bar{M}$ accepts its complement and x must be in one of the two, it follows that one of the two machines will halt and accept. If M accepts, then M' halts on state "yes." If $\bar{M}$ accepts, then M' halts on "no." $\square$