Nondeterministic Space Complexity Classes

- Let L be a language.
- Then

$$L \in \text{NSPACE}(f(n))$$

  if there is an NTM with input and output that decides L and operates within space bound $f(n)$.
- NSPACE($f(n)$) is a set of languages.
- As in the linear speedup theorem (Theorem 3 on p. 61), constant coefficients do not matter.

Graph Reachability

- Let $G(V, E)$ be a directed graph (digraph).
- REACHABILITY asks if, given nodes a and b, does G contain a path from a to b?
- Can be easily solved in polynomial time by breadth-first search.
- How about the nondeterministic space complexity?

The First Try in NSPACE($n \log n$)

```
x_1 := a; {Assume a ≠ b.}
for i = 2, 3, ..., n do
    Guess x_i ∈ {v_1, v_2, ..., v_n}; {The ith node.}
end for
for i = 2, 3, ..., n do
    if (x_{i-1}, x_i) ∉ E then
        "no";
    end if
    if x_i = b then
        "yes";
    end if
end for
"no";
```

In Fact REACHABILITY $\in$ NSPACE$(\log n)$

$x := a$;
for $i = 2, 3, \ldots, n$ **do**
    Guess $y \in \{v_1, v_2, \ldots, v_n\}$; {The next node.}
    if $(x, y) \notin E$ **then**
        "no";
    end if
    if $y = b$ **then**
        "yes";
    end if
    $x := y$;
end for
"no";

Space Analysis

- Variables i, x, and y each require $O(\log n)$ bits.
- Testing $(x, y) \in E$ is accomplished by consulting the input string with counters of $O(\log n)$ bits long.
- Hence
$$\text{REACHABILITY} \in \text{NSPACE}(\log n).$$
  - REACHABILITY with more than one terminal node also has the same complexity.
- REACHABILITY $\in$ P (p. 184).

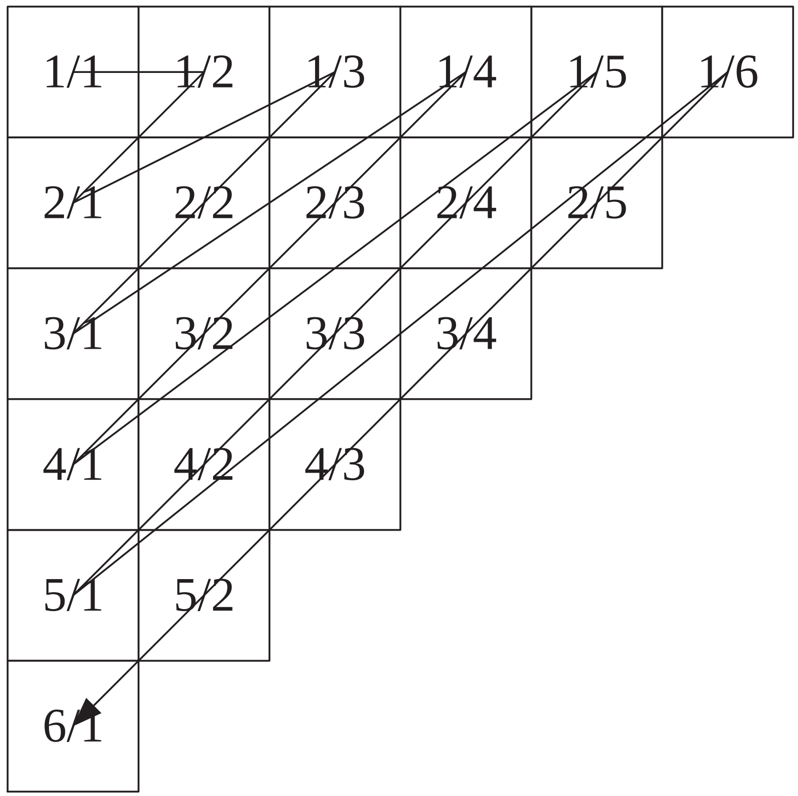
Undecidability

It seemed unworthy of a grown man
to spend his time on such trivialities,
but what was I to do?
— Bertrand Russell (1872–1970),
Autobiography, Vol. I

Infinite Sets

- A set is **countable** if it is finite or if it can be put in one-one correspondence with $\mathbb{N} = \{ 0, 1, \ldots \}$, the set of natural numbers.
  - Set of integers $\mathbb{Z}$.
    * $0 \leftrightarrow 0, 1 \leftrightarrow 1, 2 \leftrightarrow 3, 3 \leftrightarrow 5, \ldots, -1 \leftrightarrow 2, -2 \leftrightarrow 4, -3 \leftrightarrow 6, \ldots$.
  - Set of positive integers $\mathbb{Z}^+$: $i - 1 \leftrightarrow i$.
  - Set of odd integers: $(i - 1)/2 \leftrightarrow i$.
  - Set of rational numbers: See next page.

Rational Numbers Are Countable



Cardinality

- For any set A, define $|A|$ as A's **cardinality** (size).
- Two sets are said to have the same cardinality, or

$$|A| = |B| \quad \text{or} \quad A \sim B,$$

  if there exists a one-to-one correspondence between their elements.
- 2^A denotes set A's **power set**, that is $\{B : B \subseteq A\}$.
  - If $|A| = k$, then $|2^A| = 2^k$.
  - $|A| < |2^A|$ when A is finite as $k < 2^k$.

Cardinality (concluded)

- Define $|A| \leq |B|$ if there is a one-to-one correspondence between A and a subset of B's.
- Define $|A| < |B|$ if $|A| \leq |B|$ but $|A| \neq |B|$.
- Obviously, if $A \subseteq B$, then $|A| \leq |B|$.
- But if $A \subsetneq B$, then $|A| < |B|$?

Cardinality and Infinite Sets

- If A and B are infinite sets, it is possible that $A \subsetneq B$ yet $|A| = |B|$.
  - The set of integers *properly* contains the set of odd integers.
  - But the set of integers has the same cardinality as the set of odd integers (p. 96).
- A lot of "paradoxes" arise.

Galileo's[a] Paradox (1638)

- The squares of the positive integers can be placed in one-to-one correspondence with all the positive integers.
- This is contrary to the axiom of Euclid[b] that the whole is greater than any of its proper parts.
- Resolution of paradoxes: Pick the notion that results in "better" mathematics.
- The difference between a mathematical paradox and a contradiction is often a matter of opinion.

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[a] Galileo (1564–1642).
[b] Euclid (325 B.C.–265 B.C.).

Hilbert's[a] Paradox of the Grand Hotel

- For a hotel with a finite number of rooms with all the rooms occupied, a new guest will be turned away.
- Now imagine a hotel with an infinite number of rooms, all of which are occupied.
- A new guest comes and asks for a room.
- "But of course!" exclaims the proprietor.
- He moves the person previously occupying Room 1 to Room 2, the person from Room 2 to Room 3, and so on.
- The new customer now occupies Room 1.

[a] David Hilbert (1862–1943).

Hilbert's Paradox of the Grand Hotel (concluded)

- Now imagine a hotel with an infinite number of rooms, all taken up.
- An infinite number of new guests come in and ask for rooms.
- "Certainly," says the proprietor.
- He moves the occupant of Room 1 to Room 2, the occupant of Room 2 to Room 4, and so on.
- Now all odd-numbered rooms become free and the infinity of new guests can be accommodated in them.
- "There are many rooms in my Father's house, and I am going to prepare a place for you." (*John* 14:3)

David Hilbert (1862–1943)

Cantor's[a] Theorem

Theorem 6 *The set of all subsets of* $\mathbb{N}$ $(2^{\mathbb{N}})$ *is infinite and not countable.*

- Suppose it is countable with $f : \mathbb{N} \to 2^{\mathbb{N}}$ being a bijection.
- Consider the set $B = \{k \in \mathbb{N} : k \notin f(k)\} \subseteq \mathbb{N}$.
- Suppose $B = f(n)$ for some $n \in \mathbb{N}$.

---

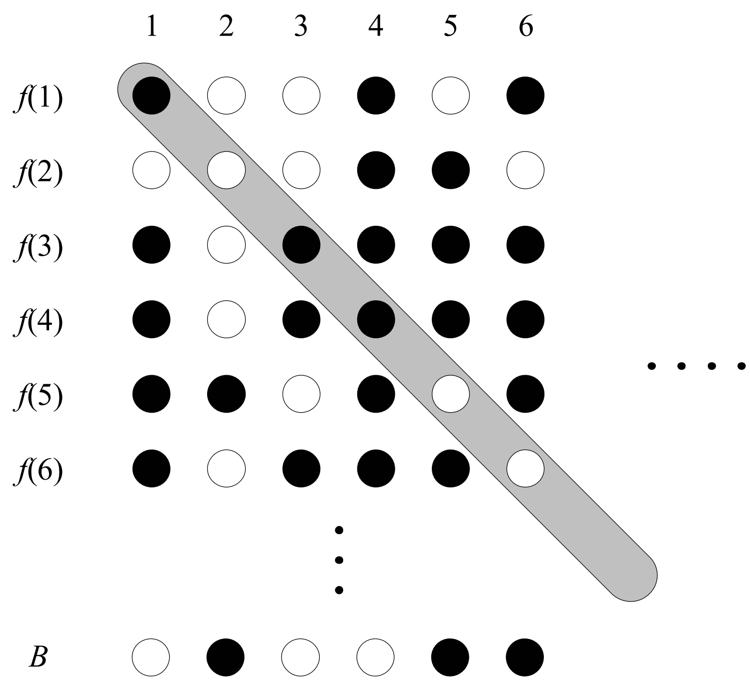
[a]Georg Cantor (1845–1918). According to Kac and Ulam, "[If] one had to name a single person whose work has had the most decisive influence on the present spirit of mathematics, it would almost surely be Georg Cantor."

The Proof (concluded)

- If $n \in f(n) = B$, then $n \in B$, but then $n \notin B$ by B's definition.
- If $n \notin f(n) = B$, then $n \notin B$, but then $n \in B$ by B's definition.
- Hence $B \neq f(n)$ for any n.
- f is not a bijection, a contradiction.

Georg Cantor (1845–1918)

Cantor's Diagonalization Argument Illustrated



A Corollary of Cantor's Theorem

Corollary 7 *For any set T, finite or infinite,*

$$|\,T\,| < |\,2^T\,|.$$

- The inequality holds in the finite T case.
- Assume T is infinite now.
- To prove $|\,T\,| \leq |2^T|$, simply consider $f(x) = \{x\} \in 2^T$.
  - f associates $T = \{\,a, b, c, \ldots\,\}$ with $\{\,\{\,a\,\}, \{\,b\,\}, \{\,c\,\}, \ldots\,\} \subseteq 2^T$.
- To prove the strict inequality $|\,T\,| \lneq |2^T|$, we use the same argument as Cantor's theorem.

A Second Corollary of Cantor's Theorem

Corollary 8 *The set of all functions on $\mathbb{N}$ is not countable.*

- It suffices to prove it for functions from $\mathbb{N}$ to $\{0,1\}$.
- Every such function $f : \mathbb{N} \to \{0,1\}$ determines a set

$$\{n : f(n) = 1\} \subseteq \mathbb{N}$$

  and vice versa.
- So the set of functions from $\mathbb{N}$ to $\{0,1\}$ has cardinality $|\, 2^{\mathbb{N}} \,|$.
- Corollary 7 (p. 109) then implies the claim.

Existence of Uncomputable Problems

- Every program is a finite sequence of 0s and 1s, thus a nonnegative integer.
- Hence every program corresponds to some integer.
- The set of programs is countable.
- A function is a mapping from integers to integers.
- The set of functions is not countable by Corollary 8 (p. 110).
- So there are functions for which no programs exist.

Universal Turing Machine[a]

- A **universal Turing machine** U interprets the input as the *description* of a TM M concatenated with the *description* of an input to that machine, x.
  - Both M and x are over the alphabet of U.
- U simulates M on x so that

$$U(M; x) = M(x).$$

- U is like a modern computer, which executes any valid machine code, or a Java Virtual machine, which executes any valid bytecode.

[a] Turing (1936).

The Halting Problem

- **Undecidable problems** are problems that have no algorithms or languages that are not recursive.
- We knew undecidable problems exist (p. 111).
- We now define a concrete undecidable problem, the **halting problem**:

$$H = \{M; x : M(x) \neq \nearrow\}.$$

  – Does M halt on input x?

H Is Recursively Enumerable

- Use the universal TM U to simulate M on x.
- When M is about to halt, U enters a "yes" state.
- If $M(x)$ diverges, so does U.
- This TM accepts H.
- Membership of x in any recursively enumerative language accepted by M can be answered by asking

$$M; x \in H?$$

H Is Not Recursive

- Suppose there is a TM M_H that *decides* H.
- Consider the program $D(M)$ that calls M_H:

```
if M_H(M; M) = "yes" then
    ↗; {Writing an infinite loop is easy, right?}
else
    "yes";
end if
```

- Consider $D(D)$:
  - $D(D) = \nearrow \Rightarrow M_H(D; D) = \text{"yes"} \Rightarrow D; D \in H \Rightarrow D(D) \neq \nearrow$, a contradiction.
  - $D(D) = \text{"yes"} \Rightarrow M_H(D; D) = \text{"no"} \Rightarrow D; D \notin H \Rightarrow D(D) = \nearrow$, a contradiction.

Comments

- Two levels of interpretations of M:
  - A sequence of 0s and 1s (data).
  - An encoding of instructions (programs).
- There are no paradoxes.
  - Concepts should be familiar to computer scientists.
  - Feed a C compiler to a C compiler, a Lisp interpreter to a Lisp interpreter, etc.

Self-Loop Paradoxes

Cantor's Paradox (1899): Let T be the set of all sets.[a]

- Then $2^T \subseteq T$ because 2^T is a set.
- But we know $|\, 2^T \,| > |\, T \,|$ (p. 109)!
- We got a "contradiction."
- So what gives?
- Are we willing to give up Cantor's theorem?
- If not, what is a set?

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[a]Recall this ontological argument for the existence of God by St Anselm (–1109) in the 11th century: If something is possible but is not part of God, then God is not the greatest possible object of thought, a contradiction.

Self-Loop Paradoxes (continued)

Russell's Paradox (1901): Consider $R = \{A : A \not\in A\}$.

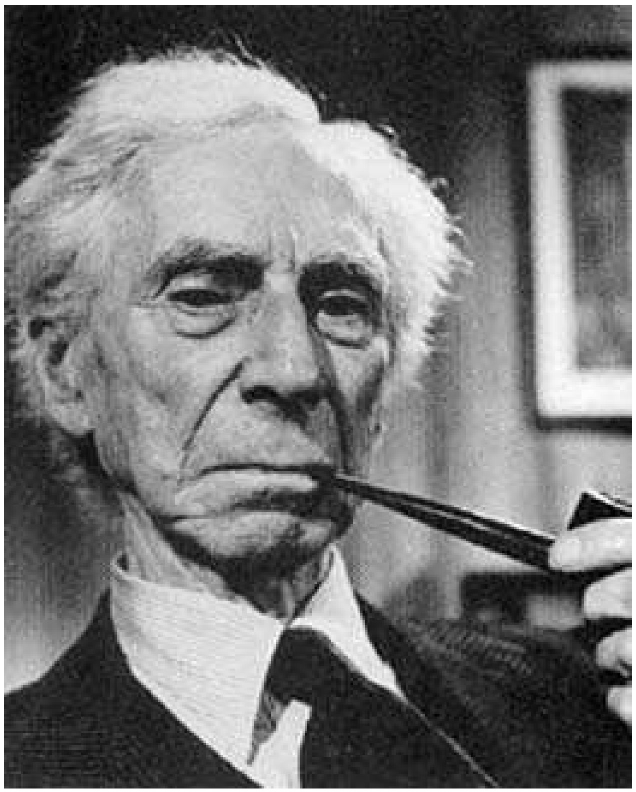
- If $R \in R$, then $R \not\in R$ by definition.
- If $R \not\in R$, then $R \in R$ also by definition.
- In either case, we have a "contradiction."

Eubulides: The Cretan says, "All Cretans are liars."

Liar's Paradox: "This sentence is false."

Hypochondriac: a patient (like Gödel) with imaginary symptoms and ailments.

Bertrand Russell (1872–1970)



Self-Loop Paradoxes (concluded)

Sharon Stone in *The Specialist* (1994): "I'm not a woman you can trust."

Spin City: "I am not gay, but my boyfriend is."

Numbers 12:3, Old Testament: "Moses was the most humble person in all the world $[\cdots]$" (attributed to Moses).

Self-Loop Paradoxes and Turing Machine?[a]

- Can self-loop paradoxes happen to Turing machine?
- If so, will it shake the foundation of the theory of computation?
- If not, why?

[a]Contributed by a student at Vanung University on June 6, 2008.

Reductions in Proving Undecidability

- Suppose we are asked to prove L is undecidable.
- Language H is known to be undecidable.
- We try to find a computable transformation (called **reduction**) R such that[a]

$$\forall x \, \{ R(x) \in L \text{ if and only if } x \in H \}.$$

- We can answer "$x \in H$?" for any x by asking "$R(x) \in L$?" instead.
- This suffices to prove that L is undecidable.

[a] Contributed by Mr. Tai-Dai Chou (J93922005) on May 19, 2005.

More Undecidability

- $H^* = \{M : M \text{ halts on all inputs}\}$.
  - Given the question "$M; x \in H$?" we construct the following machine:[a]

$$M_x(y) : M(x).$$

  - M_x halts on all inputs if and only if M halts on x.
  - In other words, $M_x \in H^*$ if and only if $M; x \in H$.
  - So if H^* were recursive, H would be recursive, a contradiction.

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[a]Simplified by Mr. Chih-Hung Hsieh (D95922003) on October 5, 2006. M_x ignores its input y; x is part of M_x's code but not M_x's input.

More Undecidability (concluded)

- $\{M; x : \text{there is a } y \text{ such that } M(x) = y\}$.
- $\{M; x : \text{the computation } M \text{ on input } x \text{ uses all states of } M\}$.
- $\{M; x; y : M(x) = y\}$.

Complements of Recursive Languages

Lemma 9 *If L is recursive, then so is $\bar{L}$.*

- Let L be decided by M (which is deterministic).
- Swap the "yes" state and the "no" state of M.
- The new machine decides $\bar{L}$.

Recursive and Recursively Enumerable Languages

Lemma 10 *L is recursive if and only if both L and $\bar{L}$ are recursively enumerable.*

- Suppose both L and $\bar{L}$ are recursively enumerable, accepted by M and $\bar{M}$, respectively.
- Simulate M and $\bar{M}$ in an *interleaved* fashion.
- If M accepts, then $x \in L$ and M' halts on state "yes."
- If $\bar{M}$ accepts, then $x \notin L$ and M' halts on state "no."

A Very Useful Corollary and Its Consequences

Corollary 11 *L is recursively enumerable but not recursive, then $\bar{L}$ is not recursively enumerable.*

- Suppose $\bar{L}$ is recursively enumerable.
- Then both L and $\bar{L}$ are recursively enumerable.
- By Lemma 10 (p. 126), L is recursive, a contradiction.

Corollary 12 *$\bar{H}$ is not recursively enumerable.*

R, RE, and coRE

RE: The set of all recursively enumerable languages.

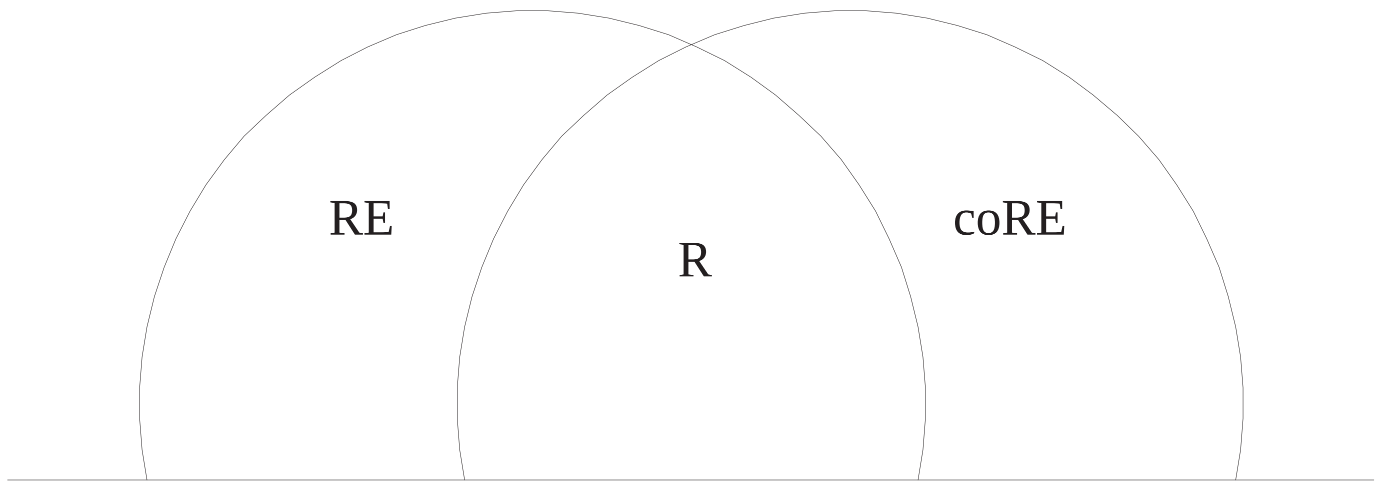
coRE: The set of all languages whose complements are recursively enumerable (note that coRE is not $\overline{\text{RE}}$).

- $\text{coRE} = \{ L : \overline{L} \in \text{RE} \}$.
- $\overline{\text{RE}} = \{ L : L \notin \text{RE} \}$.

R: The set of all recursive languages.

R, RE, and coRE (concluded)

- R $=$ RE $\cap$ coRE (p. 126).
- There exist languages in RE but not in R and not in coRE.
  - Such as H (p. 114, p. 115, and p. 127).
- There are languages in coRE but not in RE.
  - Such as $\bar{H}$ (p. 127).
- There are languages in neither RE nor coRE.


RE
R
coRE

Undecidability in Logic and Mathematics

- First-order logic is undecidable.[a]
- Natural numbers with addition and multiplication is undecidable.[b]
- Rational numbers with addition and multiplication is undecidable.[c]

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[a]Church (1936).
[b]Rosser (1937).
[c]Robinson (1948).