Theory of Computation

Homework 3
Due: 2008/11/25

Problem 1. Define MAJORITY-3-COLORING to be the problem of asking whether the nodes of a given undirected graph $G = (V, E)$ can be colored with $0, 1$ or 2 such that the following two conditions hold:

1. No two adjacent nodes have the same color.

2. At least $|V|/2$ nodes have the color 2.

Find a logarithmic-space reduction from 3-COLORING to MAJORITY-3-COLORING or prove that such reductions cannot exist.

Proof. We show a logarithmic-space reduction from 3-COLORING to MAJORITY-3-COLORING. Given an undirected graph $G = (V, E)$, the reduction adds $|V| + 1$ isolated nodes to G and outputs the resulting graph G'.

Suppose that $G \in$ 3-COLORING. Then the nodes of G can be colored with $0, 1$ or 2 such that no two adjacent nodes have the same color. By coloring the rest of the $|V| + 1$ nodes with the color 2, we see that $G' \in$ MAJORITY-3-COLORING. Conversely, $G' \in$ MAJORITY-3-COLORING clearly implies $G \in$ 3-COLORING. □

Problem 2. Let p be an odd prime and $\phi(\cdot)$ be Euler's function as in the slides. Prove or disprove that

$$\frac{|\{2i \bmod p \mid 1 \leq i \leq p\}|}{p} > \frac{\phi(3p)}{3p-1}.$$

Proof. We have

$$\{2i \bmod p \mid 1 \leq i \leq p\} = \{0, \ldots, p-1\}$$

because $2i \not\equiv 2j \bmod p$ holds for all distinct $1 \leq i, j \leq p$. As 3 is not relatively prime to $3p$, $\phi(3p) < 3p - 1$. Therefore,

$$\frac{|\{2i \bmod p \mid 1 \leq i \leq p\}|}{p} = 1 > \frac{\phi(3p)}{3p-1}.$$

□