Satisfiability

- A boolean expression ϕ is **satisfiable** if there is a truth assignment T appropriate to it such that $T \models \phi$.
- ϕ is **valid** or a **tautology**,[a] written $\models \phi$, if $T \models \phi$ for all T appropriate to ϕ.
- ϕ is **unsatisfiable** if and only if ϕ is false under all appropriate truth assignments if and only if $\neg\phi$ is valid.

[a]Wittgenstein (1889–1951) in 1922. Wittgenstein is one of the most important philosophers of all time. “God has arrived,” the great economist Keynes (1883–1946) said of him on January 18, 1928. “I met him on the 5:15 train.”

Ludwig Wittgenstein (1889–1951)

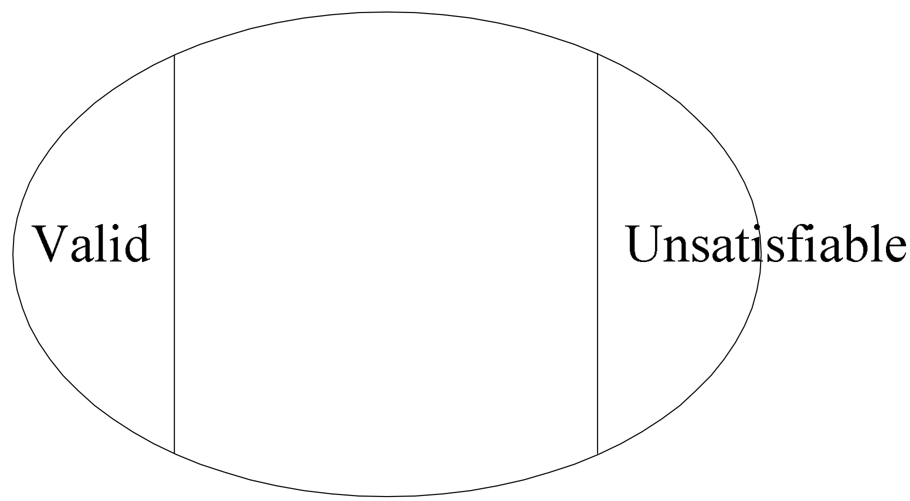
SATISFIABILITY (SAT)

- The **length** of a boolean expression is the length of the string encoding it.
- SATISFIABILITY (SAT): Given a CNF ϕ, is it satisfiable?
- Solvable in exponential time on a TM by the truth table method.
- Solvable in polynomial time on an NTM, hence in NP (p. 95).
- A most important problem in answering the $\mathrm{P} = \mathrm{NP}$ problem (p. 266).

UNSATISFIABILITY (UNSAT or SAT COMPLEMENT) and VALIDITY

- UNSAT (SAT COMPLEMENT): Given a boolean expression ϕ, is it unsatisfiable?
- VALIDITY: Given a boolean expression ϕ, is it valid?
  - ϕ is valid if and only if $\neg\phi$ is unsatisfiable.
  - So UNSAT and VALIDITY have the same complexity.
- Both are solvable in exponential time on a TM by the truth table method.

Relations among SAT, UNSAT, and VALIDITY



- The negation of an unsatisfiable expression is a valid expression.
- None of the three problems—satisfiability, unsatisfiability, validity—are known to be in P.

Boolean Functions

- An n-ary boolean function is a function

$$f : \{\texttt{true}, \texttt{false}\}^n \to \{\texttt{true}, \texttt{false}\}.$$

- It can be represented by a truth table.
- There are 2^{2^n} such boolean functions.
  - Each of the 2^n truth assignments can make f true or false.

Boolean Functions (continued)

- A boolean expression expresses a boolean function.
  - Think of its truth value under all truth assignments.
- A boolean function expresses a boolean expression.
  - $\bigvee_{T \,\models\, \phi,\ \text{literal } y_i \text{ is true under } T} (y_1 \wedge \cdots \wedge y_n)$.
    * $y_1 \wedge \cdots \wedge y_n$ is the **minterm** over $\{x_1, \ldots, x_n\}$ for T.
  - The length[a] is $\leq n2^n \leq 2^{2n}$.

[a] We count the logical connectives here.

Boolean Functions (continued)

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

The corresponding boolean expression:

$$(\neg x_1 \land \neg x_2) \lor (\neg x_1 \land x_2) \lor (x_1 \land x_2).$$

Boolean Functions (concluded)

Corollary 14 *Every n-ary boolean function can be expressed by a size-$n2^n$ boolean expression.*

In general, the exponential length in n cannot be avoided (p. 171).

Boolean Circuits

- A **boolean circuit** is a graph C whose nodes are the **gates**.
- There are no cycles in C.
- All nodes have indegree (number of incoming edges) equal to 0, 1, or 2.
- Each gate has a **sort** from

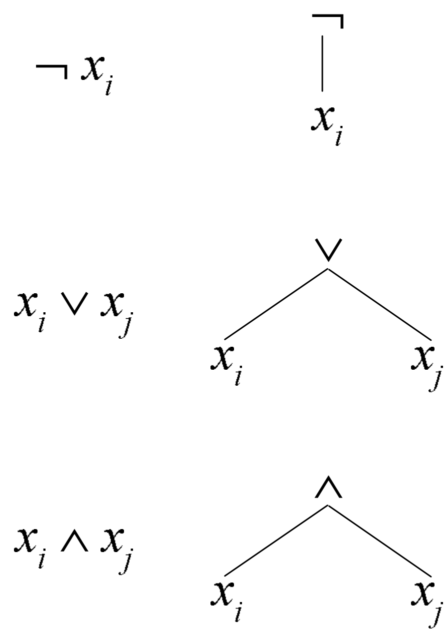
$$\{\texttt{true}, \texttt{false}, \lor, \land, \neg, x_1, x_2, \ldots\}.$$

Boolean Circuits (concluded)

- Gates of sort from $\{\texttt{true}, \texttt{false}, x_1, x_2, \ldots\}$ are the **inputs** of C and have an indegree of zero.
- The **output gate**(s) has no outgoing edges.
- A boolean circuit computes a boolean function.
- The same boolean function can be computed by infinitely many boolean circuits.

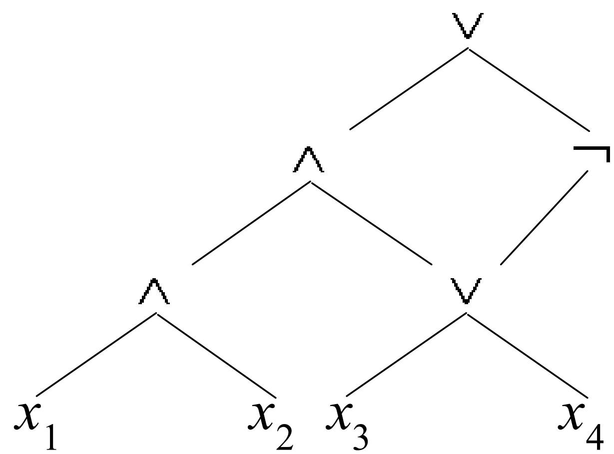
Boolean Circuits and Expressions

- They are equivalent representations.
- One can construct one from the other:



An Example

$$((x_1 \wedge x_2) \wedge (x_3 \vee x_4)) \vee (\neg (x_3 \vee x_4))$$



- Circuits are more economical because of the possibility of sharing.

CIRCUIT SAT and CIRCUIT VALUE

CIRCUIT SAT: Given a circuit, is there a truth assignment such that the circuit outputs true?

CIRCUIT VALUE: The same as CIRCUIT SAT except that the circuit has no variable gates.

- CIRCUIT SAT $\in$ NP: Guess a truth assignment and then evaluate the circuit.
- CIRCUIT VALUE $\in$ P: Evaluate the circuit from the input gates gradually towards the output gate.

Some Boolean Functions Need Exponential Circuits[a]

Theorem 15 (Shannon (1949)) *For any $n \geq 2$, there is an n-ary boolean function f such that no boolean circuits with $2^n/(2n)$ or fewer gates can compute it.*

- There are 2^{2^n} different n-ary boolean functions (see p. 162).
- So it suffices to prove that the number of boolean circuits with $2^n/(2n)$ or fewer gates is less than 2^{2^n}.

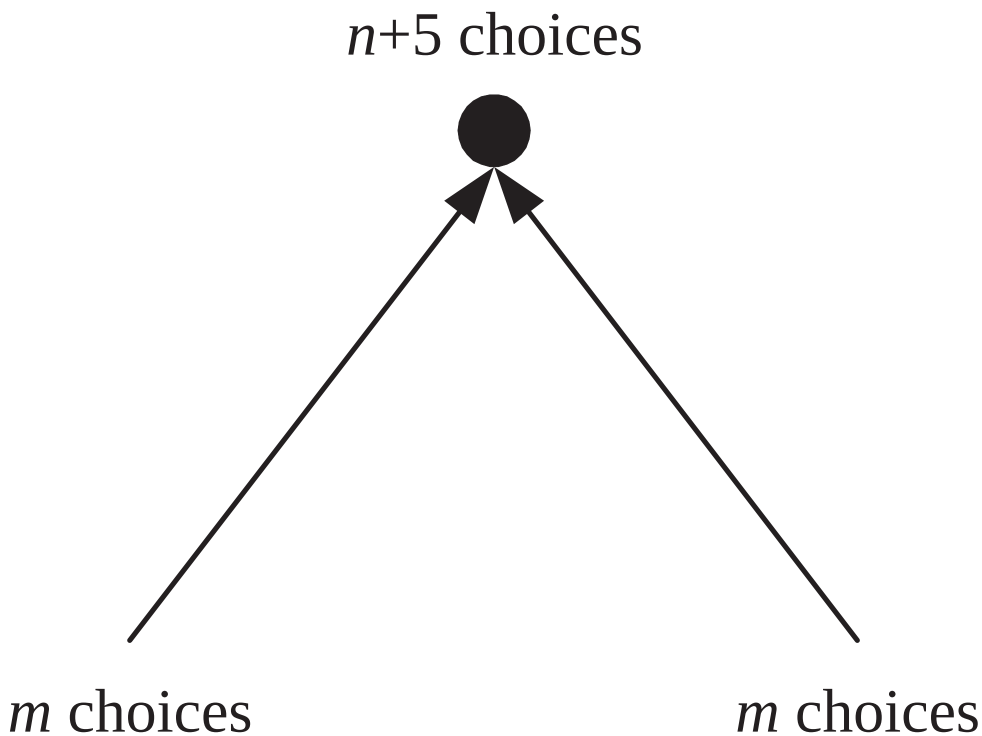
[a] Can be strengthened to "almost all boolean functions ..."

The Proof (concluded)

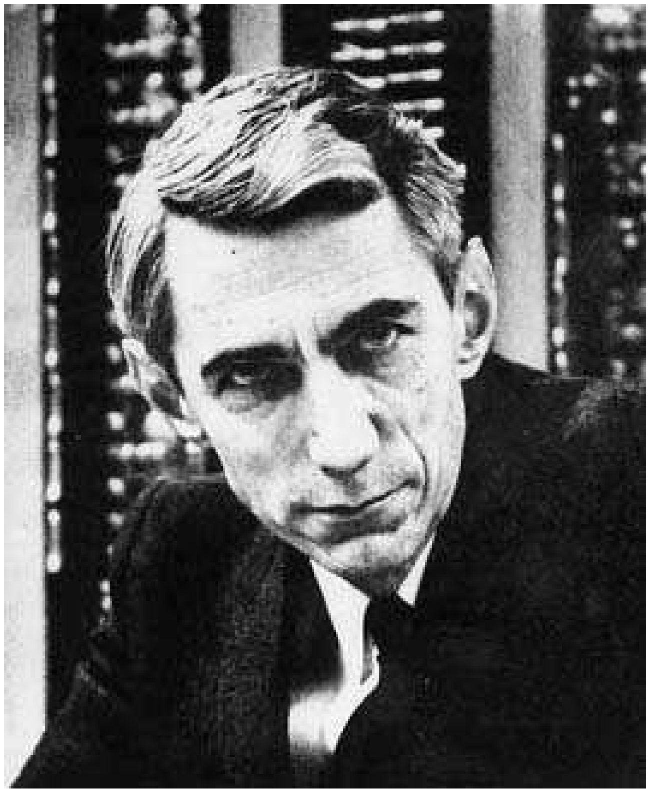
- There are at most $((n+5) \times m^2)^m$ boolean circuits with m or fewer gates (see next page).
- But $((n+5) \times m^2)^m < 2^{2^n}$ when $m = 2^n/(2n)$:

$$\begin{aligned} & m \log_2((n+5) \times m^2) \\ = \quad & 2^n \left(1 - \frac{\log_2 \frac{4n^2}{n+5}}{2n} \right) \\ < \quad & 2^n \end{aligned}$$

for $n \geq 2$.


n+5 choices
m choices
m choices

Claude Elwood Shannon (1916–2001)



Comments

- The lower bound is rather tight because an upper bound is $n2^n$ (p. 163).
- In the proof, we counted the number of circuits.
- Some circuits may not be valid at all.
- Others may compute the same boolean functions.
- Both are fine because we only need an upper bound.
- We do not need to consider the outdoing edges because they have been counted in the incoming edges.

Relations between Complexity Classes

Proper (Complexity) Functions

- We say that $f : \mathbb{N} \to \mathbb{N}$ is a **proper (complexity) function** if the following hold:
  - f is nondecreasing.
  - There is a k-string TM M_f such that $M_f(x) = \sqcap^{f(\,|\,x\,|\,)}$ for any x.[a]
  - M_f halts after $O(\,|\,x\,|\, + f(\,|\,x\,|\,))$ steps.
  - M_f uses $O(f(\,|\,x\,|\,))$ space besides its input x.
- M_f's behavior depends only on $|\,x\,|$ not x's contents.
- M_f's running time is basically bounded by $f(n)$.

---

[a]This point will become clear in Proposition 16 on p. 181.

Examples of Proper Functions

- Most "reasonable" functions are proper: c, $\lceil \log n \rceil$, polynomials of n, 2^n, $\sqrt{n}$, $n!$, etc.
- If f and g are proper, then so are $f + g$, fg, and 2^g.
- Nonproper functions when serving as the time bounds for complexity classes spoil "the theory building."
  - For example, $\text{TIME}(f(n)) = \text{TIME}(2^{f(n)})$ for some recursive function f (the **gap theorem**).[a]
- Only proper functions f will be used in $\text{TIME}(f(n))$, $\text{SPACE}(f(n))$, $\text{NTIME}(f(n))$, and $\text{NSPACE}(f(n))$.

[a]Trakhtenbrot (1964); Borodin (1972).

Space-Bounded Computation and Proper Functions

- In the definition of *space-bounded* computations, the TMs are not required to halt at all.
- When the space is bounded by a proper function f, computations can be assumed to halt:
  - Run the TM associated with f to produce an output of length $f(n)$ first.
  - The space-bound computation must repeat a configuration if it runs for more than $c^{n+f(n)}$ steps for some c (p. 198).
  - So we can count steps to prevent infinite loops.

Precise Turing Machines

- A TM M is **precise** if there are functions f and g such that for every $n \in \mathbb{N}$, for every x of length n, and for every computation path of M,
  - M halts after precisely $f(n)$ steps, and
  - All of its strings are of length precisely $g(n)$ at halting.
    * If M is a TM with input and output, we exclude the first and the last strings.
- M can be deterministic or nondeterministic.

Precise TMs Are General

Proposition 16 *Suppose a TM*[a] *M decides L within time (space) $f(n)$, where f is proper. Then there is a precise TM M' which decides L in time $O(n + f(n))$ (space $O(f(n))$, respectively).*

- M' on input x first simulates the TM M_f associated with the proper function f on x.
- M_f's output of length $f(|x|)$ will serve as a "yardstick" or an "alarm clock."

---

[a] It can be deterministic or nondeterministic.

Important Complexity Classes

- We write expressions like n^k to denote the union of all complexity classes, one for each value of k.
- For example,

$$\text{NTIME}(n^k) = \bigcup_{j>0} \text{NTIME}(n^j).$$

Important Complexity Classes (concluded)

$$\begin{aligned}
\text{P} &= \text{TIME}(n^k), \\
\text{NP} &= \text{NTIME}(n^k), \\
\text{PSPACE} &= \text{SPACE}(n^k), \\
\text{NPSPACE} &= \text{NSPACE}(n^k), \\
\text{E} &= \text{TIME}(2^{kn}), \\
\text{EXP} &= \text{TIME}(2^{n^k}), \\
\text{L} &= \text{SPACE}(\log n), \\
\text{NL} &= \text{NSPACE}(\log n).
\end{aligned}$$

Complements of Nondeterministic Classes

- From p. 139, we know R, RE, and coRE are distinct.
  - coRE contains the complements of languages in RE, *not* the languages not in RE.
- Recall that the **complement** of L, denoted by $\bar{L}$, is the language $\Sigma^* - L$.
  - SAT COMPLEMENT is the set of unsatisfiable boolean expressions.
  - HAMILTONIAN PATH COMPLEMENT is the set of graphs without a Hamiltonian path.

The Co-Classes

- For any complexity class $\mathcal{C}$, co$\mathcal{C}$ denotes the class

$$\{\bar{L} : L \in \mathcal{C}\}.$$

- Clearly, if $\mathcal{C}$ is a *deterministic* time or space *complexity class*, then $\mathcal{C} = \text{co}\mathcal{C}$.
  - They are said to be **closed under complement**.
  - A deterministic TM deciding L can be converted to one that decides $\bar{L}$ within the same time or space bound by reversing the "yes" and "no" states.
- Whether nondeterministic classes for time are closed under complement is not known (p. 87).

Comments

- Then co$\mathcal{C}$ is the class

$$\{\bar{L} : L \in \mathcal{C}\}.$$

  - So $L \in \mathcal{C}$ if and only if $\bar{L} \in \text{co}\mathcal{C}$.

- But it is *not* true that $L \in \mathcal{C}$ if and only if $L \notin \text{co}\mathcal{C}$.
  - co$\mathcal{C}$ is not defined as $\bar{\mathcal{C}}$.
- For example, suppose $\mathcal{C} = \{\{2, 4, 6, 8, 10, \ldots\}\}$.
- Then $\text{co}\mathcal{C} = \{\{1, 3, 5, 7, 9, \ldots\}\}$.
- But $\bar{\mathcal{C}} = 2^{\{1,2,3,\ldots\}^*} - \{\{2, 4, 6, 8, 10, \ldots\}\}$.

The Quantified Halting Problem

- Let $f(n) \geq n$ be proper.
- Define

$$H_f = \{M; x : M \textit{ accepts} \text{ input } x \text{ after at most } f(|\,x\,|) \text{ steps}\},$$

where M is deterministic.

- Assume the input is binary.

$H_f \in \mathsf{TIME}(f(n)^3)$

- For each input $M; x$, we simulate M on x with an alarm clock of length $f(|\, x \,|)$.
  - Use the single-string simulator (p. 66), the universal TM (p. 123), and the linear speedup theorem (p. 72).
  - Our simulator accepts $M; x$ if and only if M accepts x before the alarm clock runs out.
- From p. 71, the total running time is $O(\ell_M k_M^2 f(n)^2)$, where ℓ_M is the length to encode each symbol or state of M and k_M is M's number of strings.
- As $\ell_M k_M^2 = O(n)$, the running time is $O(f(n)^3)$, where the constant is independent of M.

$H_f \notin \mathsf{TIME}(f(\lfloor n/2 \rfloor))$

- Suppose TM M_{H_f} decides H_f in time $f(\lfloor n/2 \rfloor)$.
- Consider machine $D_f(M)$:

$$\textbf{if } M_{H_f}(M;M) = \text{“yes” } \textbf{then } \text{“no” } \textbf{else } \text{“yes”}$$

- D_f on input M runs in the same time as M_{H_f} on input $M;M$, i.e., in time $f(\lfloor \frac{2n+1}{2} \rfloor) = f(n)$, where $n = |\, M \,|$.[a]

---

[a] A student pointed out on October 6, 2004, that this estimation omits the time to write down $M;M$.

The Proof (concluded)

- First,

$$\begin{aligned} & D_f(D_f) = \text{“yes”} \\ \Rightarrow\quad & D_f; D_f \not\in H_f \\ \Rightarrow\quad & D_f \text{ does not accept } D_f \text{ within time } f(\mid D_f \mid) \\ \Rightarrow\quad & D_f(D_f) = \text{“no”} \end{aligned}$$

  a contradiction

- Similarly, $D_f(D_f) =$ “no” $\Rightarrow D_f(D_f) =$ “yes.”

The Time Hierarchy Theorem

Theorem 17 *If $f(n) \geq n$ is proper, then*

$$\text{TIME}(f(n)) \subsetneq \text{TIME}(f(2n+1)^3).$$

- The quantified halting problem makes it so.

Corollary 18 $\text{P} \subsetneq \text{EXP}$.

- $\text{P} \subseteq \text{TIME}(2^n)$ because $\text{poly}(n) \leq 2^n$ for n large enough.
- But by Theorem 17,

$$\text{TIME}(2^n) \subsetneq \text{TIME}((2^{2n+1})^3) \subseteq \text{TIME}(2^{n^2}) \subseteq \text{EXP}.$$

- So $\text{P} \subsetneq \text{EXP}$.

The Space Hierarchy Theorem

Theorem 19 (Hennie and Stearns (1966)) *If $f(n)$ is proper, then*

$$\mathrm{SPACE}(f(n)) \subsetneq \mathrm{SPACE}(f(n) \log f(n)).$$

Corollary 20 $\mathrm{L} \subsetneq \mathrm{PSPACE}$.