What Is a Proof?

- A proof convinces a party of a certain claim.
  - "Is $x^n + y^n \neq z^n$ for all $x, y, z \in \mathbb{Z}^+$ and $n > 2$?"
  - "Is graph G Hamiltonian?"
  - "Is $x^p = x \bmod p$ for prime p and $p \not| x$?"
- In mathematics, a proof is a fixed sequence of theorems.
  - Think of a written examination.
- We will extend a proof to cover a proof *process* by which the validity of the assertion is established.
  - Think of a job interview or an oral examination.

Prover and Verifier

- There are two parties to a proof.
  - The **prover** (**Peggy**).
  - The **verifier** (**Victor**).
- Given an assertion, the prover's goal is to convince the verifier of its validity (**completeness**).
- The verifier's objective is to accept only correct assertions (**soundness**).
- The verifier usually has an easier job than the prover.
- The setup is very much like the Turing test.[a]

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[a] Turing (1950).

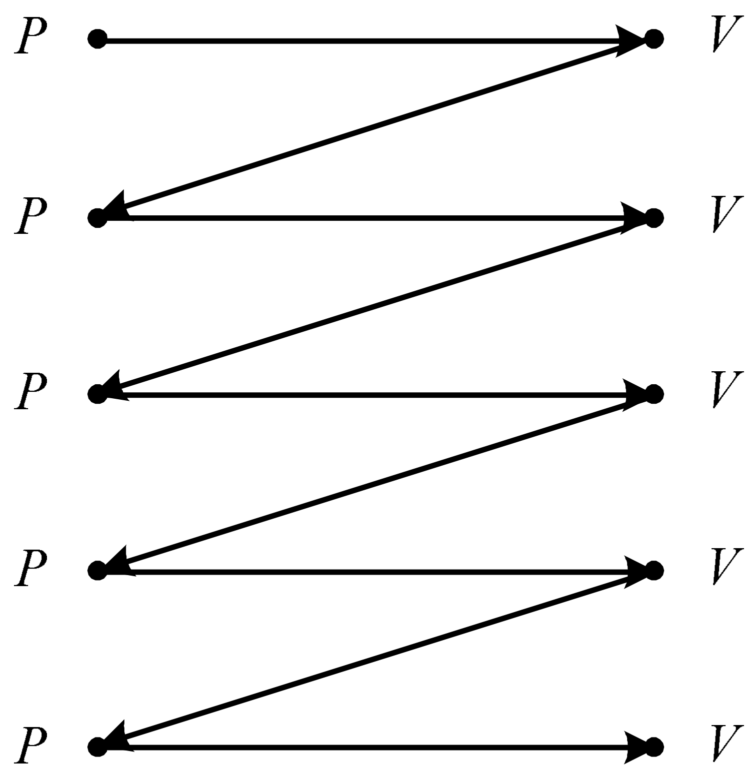
Interactive Proof Systems

- An **interactive proof** for a language L is a sequence of questions and answers between the two parties.
- At the end of the interaction, the verifier decides based on the knowledge he acquired in the proof process whether the claim is true or false.
- The verifier must be a probabilistic polynomial-time algorithm.
- The prover runs an exponential-time algorithm.
  - If the prover is not more powerful than the verifier, no interaction is needed.

Interactive Proof Systems (concluded)

- The system decides L if the following two conditions hold for any common input x.
  - If $x \in L$, then the probability that x is accepted by the verifier is at least $1 - 2^{-|x|}$.
  - If $x \notin L$, then the probability that x is accepted by the verifier with *any* prover replacing the original prover is at most $2^{-|x|}$.
- Neither the number of rounds nor the lengths of the messages can be more than a polynomial of $|x|$.

An Interactive Proof



IP[a]

- **IP** is the class of all languages decided by an interactive proof system.
- When $x \in L$, the completeness condition can be modified to require that the verifier accepts with certainty without affecting IP.[b]
- Similar things cannot be said of the soundness condition when $x \notin L$.
- Verifier's coin flips can be public.[c]

[a]Goldwasser, Micali, and Rackoff (1985).
[b]Goldreich, Mansour, and Sipser (1987).
[c]Goldwasser and Sipser (1989).

The Relations of IP with Other Classes

- NP $\subseteq$ IP.
  - IP becomes NP when the verifier is deterministic.
- BPP $\subseteq$ IP.
  - IP becomes BPP when the verifier ignores the prover's messages.
- IP actually coincides with PSPACE.[a]

[a] Shamir (1990).

Graph Isomorphism

- $V_1 = V_2 = \{1, 2, \ldots, n\}$.
- Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there exists a permutation π on $\{1, 2, \ldots, n\}$ so that $(u, v) \in E_1 \Leftrightarrow (\pi(u), \pi(v)) \in E_2$.
- The task is to answer if $G_1 \cong G_2$ (**isomorphic**).
- No known polynomial-time algorithms.
- The problem is in NP (hence IP).
- But it is not likely to be NP-complete.[a]

[a] Schöning (1987).

GRAPH NONISOMORPHISM

- $V_1 = V_2 = \{1, 2, \ldots, n\}$.
- Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **nonisomorphic** if there exist no permutations π on $\{1, 2, \ldots, n\}$ so that $(u, v) \in E_1 \Leftrightarrow (\pi(u), \pi(v)) \in E_2$.
- The task is to answer if $G_1 \not\cong G_2$ (**nonisomorphic**).
- Again, no known polynomial-time algorithms.
  - It is in coNP, but how about NP or BPP?
  - It is not likely to be coNP-complete.
- Surprisingly, GRAPH NONISOMORPHISM $\in$ IP.[a]

[a] Goldreich, Micali, and Wigderson (1986).

A 2-Round Algorithm

Victor selects a random $i \in \{1, 2\}$;
Victor selects a random permutation π on $\{1, 2, \ldots, n\}$;
Victor applies π on graph G_i to obtain graph H;
Victor sends (G_1, H) to Peggy;
if $G_1 \cong H$ **then**
    Peggy sends $j = 1$ to Victor;
else
    Peggy sends $j = 2$ to Victor;
end if
if $j = i$ **then**
    Victor accepts;
else
    Victor rejects;
end if

Analysis

- Victor runs in probabilistic polynomial time.
- Suppose $G_1 \not\cong G_2$.
  - Peggy is able to tell which G_i is isomorphic to H.
  - So Victor always accepts.
- Suppose $G_1 \cong G_2$.
  - No matter which i is picked by Victor, Peggy or any prover sees 2 identical graphs.
  - Peggy or any prover with exponential power has only probability one half of guessing i correctly.
  - So Victor erroneously accepts with probability $1/2$.
- Repeat the algorithm to obtain the desired probabilities.

Knowledge in Proofs

- Suppose I know a satisfying assignment to a satisfiable boolean expression.
- I can convince Alice of this by giving her the assignment.
- But then I give her more knowledge than necessary.
  - Alice can claim that she found the assignment!
  - Login authentication faces essentially the same issue.
  - See `www.wired.com/wired/archive/1.05/atm_pr.html` for a famous ATM fraud in the U.S.

Knowledge in Proofs (concluded)

- Digital signatures authenticate *documents* but not *individuals*.
- They hence do not solve the problem.
- Suppose I always give Alice random bits.
- Alice extracts no knowledge from me by any measure, but I prove nothing.
- Question 1: Can we design a protocol to convince Alice of (the knowledge of) a secret without revealing anything extra?
- Question 2: How to define this idea rigorously?

Zero Knowledge Proofs[a]

An interactive proof protocol (P, V) for language L has the **perfect zero-knowledge** property if:

- For every verifier V', there is an algorithm M with expected polynomial running time.
- M on any input $x \in L$ generates the same probability distribution as the one that can be observed on the communication channel of (P, V') on input x.

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[a] Goldwasser, Micali, and Rackoff (1985).

Comments

- Zero knowledge is a property of the prover.
  - It is the robustness of the prover against attempts of the verifier to extract knowledge via interaction.
  - The verifier may deviate arbitrarily (but in polynomial time) from the predetermined program.
  - A verifier cannot use the transcript of the interaction to convince a third-party of the validity of the claim.
  - The proof is hence not transferable.

Comments (continued)

- Whatever a verifier can "learn" from the specified prover P via the communication channel could as well be computed from the verifier alone.
- The verifier does not learn anything except "$x \in L$."
- Zero-knowledge proofs yield no knowledge in the sense that they can be constructed by the verifier who believes the statement, and yet these proofs do convince him.

Comments (continued)

- The "paradox" is resolved by noting that it is not the transcript of the conversation that convinces the verifier.
- But the fact that this conversation was held "on line."
- There is no zero-knowledge requirement when $x \notin L$.
- *Computational* zero-knowledge proofs are based on complexity assumptions.
  - M only needs to generate a distribution that is computationally indistinguishable from the verifier's view of the interaction.

Comments (concluded)

- It is known that if one-way functions exist, then zero-knowledge proofs exist for every problem in NP.[a]
- The verifier can be restricted to the honest one (i.e., it follows the protocol).[b]
- The coins can be public.[c]

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[a] Goldreich, Micali, and Wigderson (1986).
[b] Vadhan (2006).
[c] Vadhan (2006).

Are You Convinced?

- A newspaper commercial for hair-growing products for men.
  - A (for all practical purposes) bald man has a full head of hair after 3 months.
- A TV commercial for weight-loss products.
  - A (by any reasonable measure) overweight woman loses 10 kilograms in 10 weeks.

Quadratic Residuacity

- Let n be a product of two distinct primes.
- Assume extracting the square root of a quadratic residue modulo n is hard without knowing the factors.
- We next present a zero-knowledge proof for x being a quadratic residue.

Zero-Knowledge Proof of Quadratic Residuacity

1: **for** $m = 1, 2, \ldots, \log_2 n$ **do**

2: Peggy chooses a random $v \in Z_n^*$ and sends $y = v^2 \bmod n$ to Victor;

3: Victor chooses a random bit i and sends it to Peggy;

4: Peggy sends $z = u^i v \bmod n$, where u is a square root of x; $\{u^2 \equiv x \bmod n.\}$

5: Victor checks if $z^2 \equiv x^i y \bmod n$;

6: **end for**

7: Victor accepts x if Line 5 is confirmed every time;

Analysis

- Suppose x is a quadratic nonresidue.
  - Peggy can answer only one of the two possible challenges.
    * Reason: a is a quadratic residue if and only if xa is a quadratic nonresidue.
  - So Peggy will be caught in any given round with probability one half.

Analysis (continued)

- Suppose x is a quadratic residue.
  - Peggy can answer all challenges.
  - So Victor will accept x.
- How about the claim of zero knowledge?
- The transcript between Peggy and Victor when x is a quadratic residue can be generated without Peggy!
  - So interaction with Peggy is useless.
- Here is how.

Analysis (continued)

- Suppose x is a quadratic residue.[a]
- In each round of interaction with Peggy, the transcript is a triplet (y, i, z).
- We present an efficient Bob that generates (y, i, z) with the same probability *without* accessing Peggy.

[a]By definition, we do not need to consider the other case.

Analysis (concluded)

1: Bob chooses a random $z \in Z_n^*$;

2: Bob chooses a random bit i;

3: Bob calculates $y = z^2 x^{-i} \bmod n$;

4: Bob writes (y, i, z) into the transcript;

Comments

- Assume x is a quadratic residue.
- In both cases, for (y, i, z), y is a random quadratic residue, i is a random bit, and z is a random number.
- Bob cheats because (y, i, z) is *not* generated in the same order as in the original transcript.
  - Bob picks Victor's challenge first.
  - Bob then picks Peggy's answer.
  - Bob finally patches the transcript.

Comments (concluded)

- So it is not the transcript that convinces Victor, but that conversation with Peggy is held "on line."
- The same holds even if the transcript was generated by a cheating Victor's interaction with (honest) Peggy.
- But we skip the details.

A Useful Corollary

Corollary 76 *Let $n = pq$ be a product of two distinct primes. Then $xy \in Z_n^*$ is a quadratic residue modulo n if and only if x and y are both quadratic residues or quadratic nonresidues modulo n.*

- By Lemma 75 (p. 569), xy is a quadratic residue if and only if $(xy \,|\, p) = (xy \,|\, q) = 1$.
- This holds if and only if $(x \,|\, p)(y \,|\, p) = (x \,|\, q)(y \,|\, q) = 1$.

The Proof (concluded)

- Now,

$$(x \mid p)(y \mid p) = (x \mid q)(y \mid q) = 1$$

  if and only if

$$(x \mid p)(x \mid q) = (y \mid p)(y \mid q) = 1$$

  because Legendre symbols are ± 1.

- But the above holds if and only if x and y are both quadratic residues or quadratic nonresidues modulo n, again by Lemma 75.

Does the Following Work, Too?[a]

1: **for** $m = 1, 2, \ldots, \log_2 n$ **do**
2: Peggy chooses a random $v \in Z_n^*$ and sends $y = v^2 \bmod n$ to Victor;
3: Peggy sends $z = uv \bmod n$, where u is a square root of x; $\{u^2 \equiv x \bmod n.\}$
4: Victor checks if $z^2 \equiv xy \bmod n$;
5: **end for**
6: Victor accepts x if Line 4 is confirmed every time;

[a]Contributed by Mr. Chih-Duo Hong (`R95922079`) on December 13, 2006. It is like choosing $i = 1$ in the original protocol.

Does the Following Work, Too? (concluded)

- Suppose x is a quadratic nonresidue.
- But Peggy can mislead Victor.
- Peggy first chooses a quadratic nonresidue y.
- She can solve $z^2 = xy \bmod n$ (see Corollary 76 on p. 601).
- Finally, she sends y and z to Victor.
- This pair will satisfy $z^2 \equiv xy \bmod n$ by construction.
- The protocol is hence not even an IP protocol!

Zero-Knowledge Proof of 3 Colorability[a]

for $i = 1, 2, \ldots, |E|^2$ **do**

  Peggy chooses a random permutation π of the 3-coloring ϕ;

  Peggy samples an encryption scheme randomly and sends $\pi(\phi(1)), \pi(\phi(2)), \ldots, \pi(\phi(|V|))$ encrypted to Victor;

  Victor chooses at random an edge $e \in E$ and sends it to Peggy for the coloring of the endpoints of e;

  if $e = (u, v) \in E$ **then**

    Peggy reveals the coloring of u and v and "proves" that they correspond to their encryption;

  else

    Peggy stops;

  end if

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[a] Goldreich, Micali, and Wigderson (1986).

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10:    if the "proof" provided in Line 6 is not valid then
11:       Victor rejects and stops;
12:    end if
13:    if π(φ(u)) = π(φ(v)) or π(φ(u)), π(φ(v)) ∉ {1, 2, 3} then
14:       Victor rejects and stops;
15:    end if
16: end for
17: Victor accepts;
```

Analysis

- If the graph is 3-colorable and both Peggy and Victor follow the protocol, then Victor always accepts.
- If the graph is not 3-colorable and Victor follows the protocol, then however Peggy plays, Victor will accept with probability $\leq (1 - m^{-1})^{m^2} \leq e^{-m}$, where $m = |\, E \,|$.
- Thus the protocol is valid.
- This protocol yields no knowledge to Victor as all he gets is a bunch of random pairs.
- The proof that the protocol is zero-knowledge to *any* verifier is intricate.

Comments

- Each $\pi(\phi(i))$ is encrypted by a different cryptosystem.[a]
  - Otherwise, all the colors will be revealed in Step 6.
- Each edge e must be picked randomly.[b]
  - Otherwise, Peggy will know Victor's game plan and plot accordingly.

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[a]Contributed by Ms. Yui-Huei Chang (R96922060) on May 22, 2008

[b]Contributed by Ms. Chang-Rong Hung (R96922028) on May 22, 2008