Theory of Computation

Solutions to Homework 3

Problem 1. Show that there exist a constant $c > 0$ and a language $L \notin \text{NTIME}(n^c)$ such that L is logspace reducible to a language in $\text{NTIME}(n^c)$. You may use the nondeterministic time hierarchy theorem, proved by Cook in 1972, which implies $\text{NTIME}(n^a) \subsetneq \text{NTIME}(n^b)$ for all $b > a > 1$. (Hint: The Cook-Levin theorem states that every language in NP is logspace reducible to SAT, which lies in $\text{NTIME}(n^c)$ for some constant $c > 0$. The nondeterministic time hierarchy theorem guarantees the nonemptiness of $\text{NP} \setminus \text{NTIME}(n^c)$.)

Proof. Let $c > 0$ satisfy $\text{SAT} \in \text{NTIME}(n^c)$. By the NP-hardness of SAT, every language in $\text{NP} \setminus \text{NTIME}(n^c)$ is logspace reducible to SAT. The nondeterministic time hierarchy theorem implies $\text{NP} \setminus \text{NTIME}(n^c) \neq \emptyset$, which completes the proof. □

Problem 2. Prove that

$$\left\{ x_1, \ldots, x_n, w \in \mathbb{N} \mid \exists S \subseteq \{1, \ldots, n\} \text{ such that } \sum_{i \in S} x_i = w \geq \frac{\sum_{i=1}^{n} x_i}{2} \right\}$$

is NP-complete. You may use reductions from any problem shown to be NP-complete in class or in the textbook. For example, the following problem is shown to be NP-complete on pages 349–355 of the slides:

> Given positive integers $v_1, \ldots, v_n, K$, does there exist a subset of $\{v_1, \ldots, v_n\}$ that adds up to exactly K?

Proof. Clearly, the language whose NP-completeness is to be shown lies in NP. To show its NP-hardness, we describe a logspace reduction from EXACT COVER BY 3-SETS as in the slides. The reduction is given a family $F = \{S_1, \ldots, S_n\}$ of size-3 subsets of $U = \{1, \ldots, 3m\}$. For $1 \leq i \leq n$, let x_i be the natural number whose $(n+1)$-ary representation has a 1 on the least significant j-th bit for each $j \in S_i$, and 0's on all other digits. The output of the reduction consists of the numbers $x_1, \ldots, x_n$ and $w = \max\left\{K, \left(\sum_{i=1}^{n} x_i\right) - K\right\}$, where K is $1 \ldots 1$ in base $n+1$. Now the following three statements are equivalent.

1. There exists an $S \subseteq \{1, \ldots, n\}$ with $\sum_{i \in S} x_i = w$.
2. There exists an $S \subseteq \{1, \ldots, n\}$ with $\sum_{i \in S} x_i = K$.

3. $(S_1, \ldots, S_n, U)$ constitutes a yes-instance of EXACT COVER BY 3-SETS.

Above, items 1–2 are equivalent by the definition of w; items 2–3 are equivalent because no carry in base $n+1$ could occur when adding any numbers among $x_1, \ldots, x_n$. Finally, the proof for the correctness of the reduction is complete by the equivalence of items 1 and 3 and the trivial fact that $w \geq \sum_{i=1}^{n} x_i/2$. □