# Theory of Computation

## Solutions to Homework 2

**Problem 1.** Let $L \subseteq \{0,1\}^n$ be a non-recursive language. Define $L' = \{0x \mid x \in L\}$ where $0x$ denotes the concatenation of 0 and $x$. Show that $L'$ is non-recursive.

*Proof.* Assume for contradiction that $L'$ is recursive. Let $M$ be the Turing machine which accepts its input $x \in \{0,1\}^*$ if $0x \in L'$ and rejects it otherwise. Then $M$ clearly decides $L$, a contradiction. □

**Comment 1.** *There is a typo in the first line of the problem statement, which should be "Let $L \subseteq \{0,1\}^*$ be ..." rather than "Let $L \subseteq \{0,1\}^n$ be ..." Since we have a false premise that $L \subseteq \{0,1\}^n$, being finite, is non-recursive, the implication in the problem statement is trivially true. Answers that point out the falsity of the premise will be considered correct (as they are correct). Sorry for the typo.*

**Problem 2.** Let $L \subseteq \{0,1\}^*$ be a recursive language satisfying $|L \cap \{0,1\}^n| = 2$ for each $n \in \mathbb{N}$. Prove the existence of a non-recursive language $L' \subseteq L$. (Hint: You may want to show that $L$ has uncountably many subsets. Any other method is also welcomed.)

*Proof.* Write $L = \{x_k \mid k \in \mathbb{N}\}$ where $x_i \neq x_j$ for $i \neq j$. Assume for contradiction that $L$ has only countably many subsets $L_h$, $h \in \mathbb{N}$. Then the subset $\hat{L} = \{x_k \mid x_k \notin L_k, k \in \mathbb{N}\}$ of $L$ must equal $L_t$ for some $t \in \mathbb{N}$. But $x_t \in \hat{L}$ and $x_t \notin \hat{L}$ imply each other, which is absurd.

We have shown that $L$ has uncountably many subsets. Now since there are only countably many Turing machines and each Turing machine decides at most one language, there exists a non-recursive subset of $L$. □