MIN CUT and MAX CUT

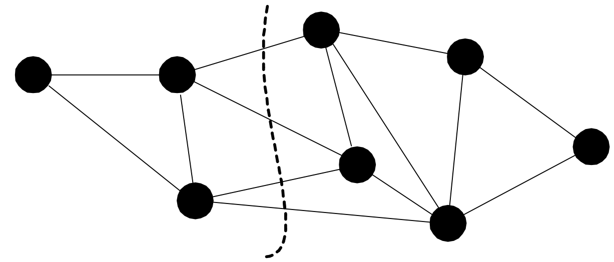
- A **cut** in an undirected graph $G = (V, E)$ is a partition of the nodes into two nonempty sets S and $V - S$.
- The size of a cut $(S, V - S)$ is the number of edges between S and $V - S$.
- MIN CUT $\in$ P by the maxflow algorithm.
- MAX CUT asks if there is a cut of size at least K.
  - K is part of the input.

MIN CUT and MAX CUT (concluded)

- MAX CUT has applications in VLSI layout.
  - The minimum area of a VLSI layout of a graph is not less than the square of its maximum cut size.[a]

[a] Raspaud, Sýkora, and Vrťo (1995).

A Cut



MAX CUT Is NP-Complete[a]

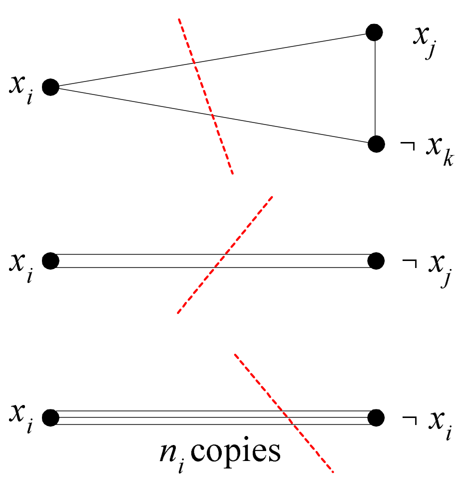
- We will reduce NAESAT to MAX CUT.
- Given an instance ϕ of 3SAT with m clauses, we shall construct a graph $G = (V, E)$ and a goal K such that:
  - There is a cut of size at least K if and only if ϕ is NAE-satisfiable.
- Our graph will have multiple edges between two nodes.
  - Each such edge contributes one to the cut if its nodes are separated.

[a] Garey, Johnson, and Stockmeyer (1976).

The Proof

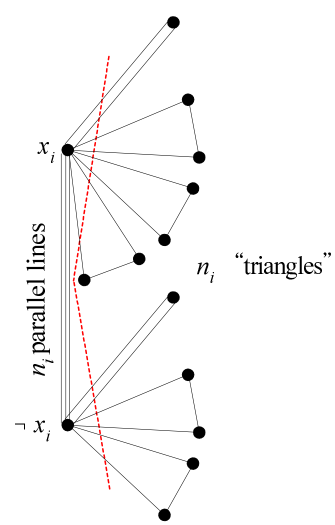
- Suppose ϕ's m clauses are $C_1, C_2, \ldots, C_m$.
- The boolean variables are $x_1, x_2, \ldots, x_n$.
- G has $2n$ nodes: $x_1, x_2, \ldots, x_n, \neg x_1, \neg x_2, \ldots, \neg x_n$.
- Each clause with 3 distinct literals makes a triangle in G.
- For each clause with two identical literals, there are two parallel edges between the two distinct literals.
- No need to consider clauses with one literal (why?).
- For each variable x_i, add n_i copies of edge $[x_i, \neg x_i]$, where n_i is the number of occurrences of x_i and $\neg x_i$ in ϕ.[a]

[a] Regardless of whether both x_i and $\neg x_i$ occur in ϕ.



The Proof (continued)

- Set $K = 5m$.
- Suppose there is a cut $(S, V - S)$ of size $5m$ or more.
- A clause (a triangle or two parallel edges) contributes at most 2 to a cut no matter how you split it.
- Suppose both x_i and $\neg x_i$ are on the same side of the cut.
- Then they *together* contribute at most $2n_i$ edges to the cut as they appear in at most n_i different clauses.

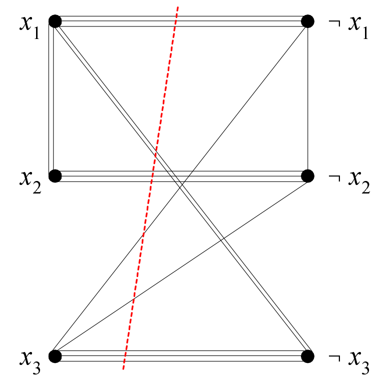


The Proof (continued)

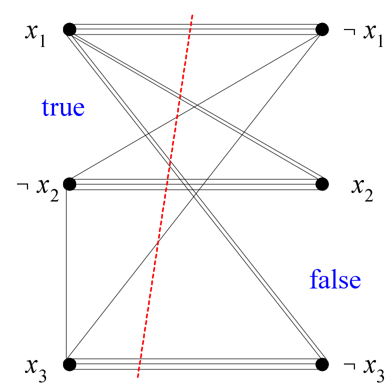
- Changing the side of a literal contributing at most n_i to the cut does not decrease the size of the cut.
- Hence we assume variables are separated from their negations.
- The total number of edges in the cut that join opposite literals is $\sum_i n_i = 3m$.
  - The total number of literals is $3m$.

The Proof (concluded)

- The *remaining* $2m$ edges in the cut must come from the m triangles or parallel edges that correspond to the clauses.
- As each can contribute at most 2 to the cut, all are split.
- A split clause means at least one of its literals is true and at least one false.
- The other direction is left as an exercise.



- $(x_1 \lor x_2 \lor x_2) \land (x_1 \lor \neg x_3 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor x_3)$.
- The cut size is $13 < 5 \times 3 = 15$.



- $(x_1 \lor x_2 \lor x_2) \land (x_1 \lor \neg x_3 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor x_3)$.
- The cut size is now 15.

A Remark

- We had proved that MAX CUT is NP-complete for multigraphs.
- How about proving the same thing for simple graphs?[a]
- For 4SAT, how do you modify the proof?[b]

[a]Contributed by Mr. Tai-Dai Chou (J93922005) on June 2, 2005.
[b]Contributed by Mr. Chien-Lin Chen (J94922015) on June 8, 2006.

MAX BISECTION

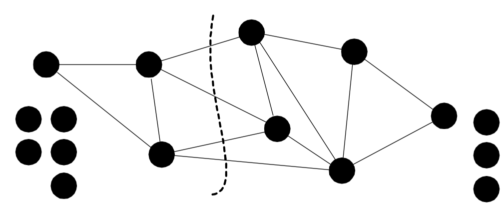
- MAX CUT becomes MAX BISECTION if we require that $|S| = |V - S|$.
- It has many applications, especially in VLSI layout.

MAX BISECTION Is NP-Complete

- We shall reduce the more general MAX CUT to MAX BISECTION.
- Add $|V|$ **isolated nodes** to G to yield G'.
- G' has $2 \times |V|$ nodes.
- As the new nodes have no edges, moving them around contributes nothing to the cut.

The Proof (concluded)

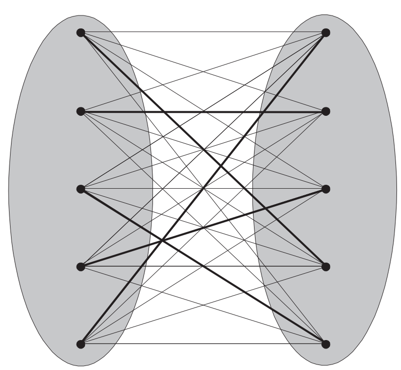
- Every cut $(S, V - S)$ of $G = (V, E)$ can be made into a bisection by appropriately allocating the new nodes between S and $V - S$.
- Hence each cut of G can be made a cut of G' of the same size, and vice versa.



BISECTION WIDTH

- BISECTION WIDTH is like MAX BISECTION except that it asks if there is a bisection of size *at most* K (sort of MIN BISECTION).
- Unlike MIN CUT, BISECTION WIDTH remains NP-complete.
  - A graph $G = (V, E)$, where $|V| = 2n$, has a bisection of size K if and only if the complement of G has a bisection of size $n^2 - K$.
  - So G has a bisection of size $\geq K$ if and only if its complement has a bisection of size $\leq n^2 - K$.

Illustration



HAMILTONIAN PATH Is NP-Complete[a]

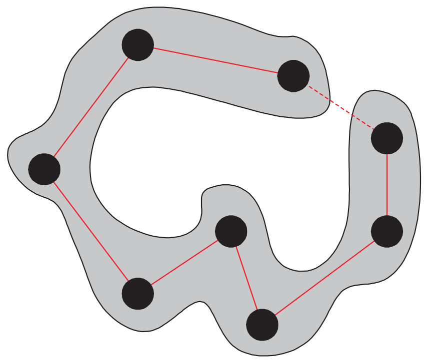
Theorem 16 *Given an* undirected *graph, the question whether it has a Hamiltonian path is NP-complete.*

[a] Karp (1972).

TSP (D) Is NP-Complete

Corollary 17 TSP (D) *is NP-complete.*

- Consider a graph G with n nodes.
- Define $d_{ij} = 1$ if $[i, j] \in G$ and $d_{ij} = 2$ if $[i, j] \notin G$.
- Set the budget $B = n + 1$.
- Suppose G has no Hamiltonian paths.
- Then every tour on the new graph must contain at least two edges with weight 2.
  - Otherwise, by removing up to one edge with weight 2, one obtains a Hamiltonian path, a contradiction.



TSP (D) Is NP-Complete (concluded)

- The total cost is then at least $(n-2)+2 \cdot 2 = n+2 > B$.
- On the other hand, suppose G has Hamiltonian paths.
- Then there is a tour on the new graph containing at most one edge with weight 2.
- The total cost is then at most $(n-1)+2 = n+1 = B$.
- We conclude that there is a tour of length B or less if and only if G has a Hamiltonian path.

Graph Coloring

- k-COLORING asks if the nodes of a graph can be colored with $\leq k$ colors such that no two adjacent nodes have the same color.
- 2-COLORING is in P (why?).
- But 3-COLORING is NP-complete (see next page).
- k-COLORING is NP-complete for $k \geq 3$ (why?).

3-COLORING Is NP-Complete[a]

- We will reduce NAESAT to 3-COLORING.
- We are given a set of clauses $C_1, C_2, \ldots, C_m$ each with 3 literals.
- The boolean variables are $x_1, x_2, \ldots, x_n$.
- We shall construct a graph G such that it can be colored with colors $\{0, 1, 2\}$ if and only if all the clauses can be NAE-satisfied.

[a] Karp (1972).

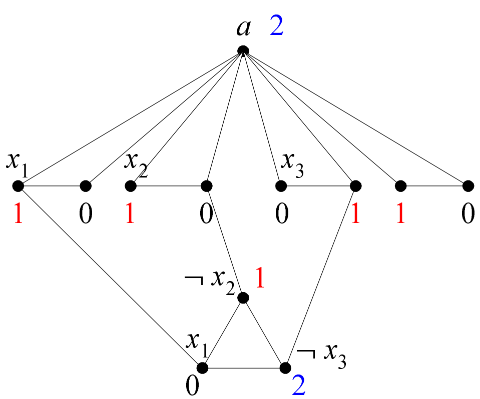
The Proof (continued)

- Every variable x_i is involved in a triangle $[\,a, x_i, \neg x_i\,]$ with a common node a.
- Each clause $C_i = (c_{i1} \vee c_{i2} \vee c_{i3})$ is also represented by a triangle

$$[\,c_{i1}, c_{i2}, c_{i3}\,].$$

  - Node c_{ij} with the same label as one in some triangle $[\,a, x_k, \neg x_k\,]$ represent *distinct* nodes.
- There is an edge between c_{ij} and the node that represents the jth literal of C_i.

Construction for $\cdots \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge \cdots$



The Proof (continued)

Suppose the graph is 3-colorable.

- Assume without loss of generality that node a takes the color 2.
- A triangle must use up all 3 colors.
- As a result, one of x_i and $\neg x_i$ must take the color 0 and the other 1.

The Proof (continued)

- Treat 1 as `true` and 0 as `false`.[a]
  - We were dealing only with those triangles with the a node, not the clause triangles.
- The resulting truth assignment is clearly contradiction free.
- As each clause triangle contains one color 1 and one color 0, the clauses are NAE-satisfied.

[a] The opposite also works.

The Proof (continued)

Suppose the clauses are NAE-satisfiable.

- Color node a with color 2.
- Color the nodes representing literals by their truth values (color 0 for `false` and color 1 for `true`).
  - We were dealing only with those triangles with the a node, not the clause triangles.

The Proof (concluded)

- For each clause triangle:
  - Pick any two literals with opposite truth values.
  - Color the corresponding nodes with 0 if the literal is `true` and 1 if it is `false`.
  - Color the remaining node with color 2.
- The coloring is legitimate.
  - If literal w of a clause triangle has color 2, then its color will never be an issue.
  - If literal w of a clause triangle has color 1, then it must be connected up to literal w with color 0.
  - If literal w of a clause triangle has color 0, then it must be connected up to literal w with color 1.

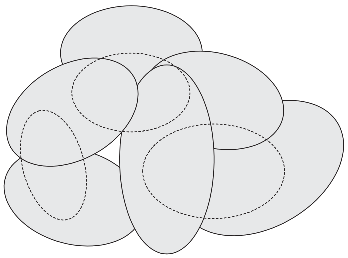
TRIPARTITE MATCHING

- We are given three sets B, G, and H, each containing n elements.
- Let $T \subseteq B \times G \times H$ be a ternary relation.
- TRIPARTITE MATCHING asks if there is a set of n triples in T, none of which has a component in common.
  - Each element in B is matched to a different element in G and different element in H.

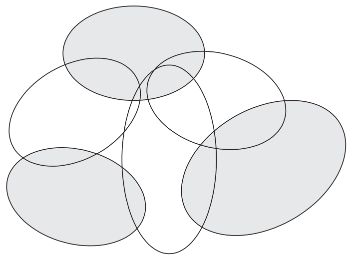
Theorem 18 (Karp (1972)) TRIPARTITE MATCHING *is NP-complete.*

Related Problems

- We are given a family $F = \{S_1, S_2, \ldots, S_n\}$ of subsets of a finite set U and a budget B.
- SET COVERING asks if there exists a set of B sets in F whose union is U.
- SET PACKING asks if there are B disjoint sets in F.
- Assume $|U| = 3m$ for some $m \in \mathbb{N}$ and $|S_i| = 3$ for all i.
- EXACT COVER BY 3-SETS asks if there are m sets in F that are disjoint and have U as their union.



SET COVERING



SET PACKING

Related Problems (concluded)

Corollary 19 SET COVERING, SET PACKING, *and* EXACT COVER BY 3-SETS *are all NP-complete.*

The KNAPSACK Problem

- There is a set of n items.
- Item i has value $v_i \in \mathbb{Z}^+$ and weight $w_i \in \mathbb{Z}^+$.
- We are given $K \in \mathbb{Z}^+$ and $W \in \mathbb{Z}^+$.
- KNAPSACK asks if there exists a subset $S \subseteq \{1, 2, \ldots, n\}$ such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i \geq K$.
  - We want to achieve the maximum satisfaction within the budget.

KNAPSACK Is NP-Complete

- KNAPSACK $\in$ NP: Guess an S and verify the constraints.
- We assume $v_i = w_i$ for all i and $K = W$.
- KNAPSACK now asks if a subset of $\{v_1, v_2, \ldots, v_n\}$ adds up to exactly K.
  - Picture yourself as a radio DJ.
  - Or a person trying to control the calories intake.
- We shall reduce EXACT COVER BY 3-SETS to KNAPSACK.

The Proof (continued)

- We are given a family $F = \{S_1, S_2, \ldots, S_n\}$ of size-3 subsets of $U = \{1, 2, \ldots, 3m\}$.
- EXACT COVER BY 3-SETS asks if there are m disjoint sets in F that cover the set U.
- Think of a set as a bit vector in $\{0, 1\}^{3m}$.
  - `001100010` means the set $\{3, 4, 8\}$, and `110010000` means the set $\{1, 2, 5\}$.
- Our goal is $\overbrace{\texttt{11}\cdots\texttt{1}}^{3m}$.

The Proof (continued)

- A bit vector can also be considered as a binary *number*.
- Set union resembles addition.
  - $\texttt{001100010} + \texttt{110010000} = \texttt{111110010}$, which denotes the set $\{1, 2, 3, 4, 5, 8\}$, as desired.
- Trouble occurs when there is *carry*.
  - $\texttt{001100010} + \texttt{001110000} = \texttt{010010010}$, which denotes the set $\{2, 5, 8\}$, not the desired $\{3, 4, 5, 8\}$.

The Proof (continued)

- Carry may also lead to a situation where we obtain our solution $\texttt{11}\cdots\texttt{1}$ with more than m sets in F.
  - $\texttt{001100010} + \texttt{001110000} + \texttt{101100000} + \texttt{000001101} = \texttt{111111111}$.
  - But this "solution" $\{1, 3, 4, 5, 6, 7, 8, 9\}$ does not correspond to an exact cover.
  - And it uses 4 sets instead of the required 3.[a]
- To fix this problem, we enlarge the base just enough so that there are no carries.
- Because there are n vectors in total, we change the base from 2 to $n + 1$.

[a] Thanks to a lively class discussion on November 20, 2002.

The Proof (continued)

- Set v_i to be the $(n + 1)$-ary number corresponding to the bit vector encoding S_i.
- Now in base $n + 1$, if there is a set S such that $\sum_{v_i \in S} v_i = \overbrace{11\cdots1}^{3m}$, then every bit position must be contributed by exactly one v_i and $|S| = m$.
- Finally, set

$$K = \sum_{j=0}^{3m-1} (n + 1)^j = \overbrace{11\cdots1}^{3m} \quad \text{(base } n + 1\text{)}.$$

The Proof (continued)

- Suppose F admits an exact cover, say $\{S_1, S_2, \ldots, S_m\}$.
- Then picking $S = \{v_1, v_2, \ldots, v_m\}$ clearly results in

$$v_1 + v_2 + \cdots + v_m = \overbrace{11 \cdots 1}^{3m}.$$

  - It is important to note that the meaning of addition (+) is independent of the base.[a]
  - It is just regular addition.

[a]Contributed by Mr. Kuan-Yu Chen (R92922047) on November 3, 2004.

The Proof (concluded)

- On the other hand, suppose there exists an S such that $\sum_{v_i \in S} v_i = \overbrace{11 \cdots 1}^{3m}$ in base $n+1$.
- The no-carry property implies that $|S| = m$ and $\{S_i : v_i \in S\}$ is an exact cover.

An Example

- Let $m = 3$, $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and

$$\begin{aligned} S_1 &= \{1, 3, 4\}, \\ S_2 &= \{2, 3, 4\}, \\ S_3 &= \{2, 5, 6\}, \\ S_4 &= \{6, 7, 8\}, \\ S_5 &= \{7, 8, 9\}. \end{aligned}$$

- Note that $n = 5$, as there are 5 S_i's.

An Example (concluded)

- Our reduction produces

$$\begin{aligned} K &= \sum_{j=0}^{3 \times 3 - 1} 6^j = \overbrace{11 \cdots 1}^{3 \times 3} \quad \text{(base 6)}, \\ v_1 &= \texttt{101100000}, \\ v_2 &= \texttt{011100000}, \\ v_3 &= \texttt{010011000}, \\ v_4 &= \texttt{000001110}, \\ v_5 &= \texttt{000000111}. \end{aligned}$$

- Note $v_1 + v_3 + v_5 = K$.
- Indeed, $S_1 \cup S_3 \cup S_5 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, an exact cover by 3-sets.

BIN PACKINGS

- We are given N positive integers $a_1, a_2, \ldots, a_N$, an integer C (the capacity), and an integer B (the number of bins).
- BIN PACKING asks if these numbers can be partitioned into B subsets, each of which has total sum at most C.
- Think of packing bags at the check-out counter.

Theorem 20 BIN PACKING *is NP-complete.*

Finis