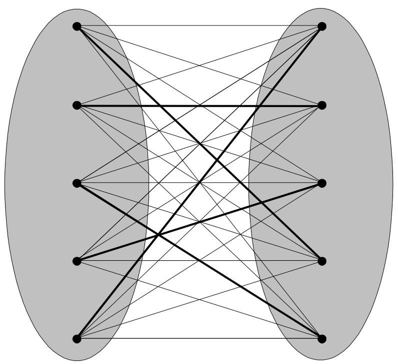
## BISECTION WIDTH

- BISECTION WIDTH is like MAX BISECTION except that it asks if there is a bisection of size *at most* $K$ (sort of MIN BISECTION).
- Unlike MIN CUT, BISECTION WIDTH remains NP-complete.
  - A graph $G = (V, E)$, where $|V| = 2n$, has a bisection of size $K$ if and only if the complement of $G$ has a bisection of size $n^2 - K$.
  - So $G$ has a bisection of size $\geq K$ if and only if its complement has a bisection of size $\leq n^2 - K$.



## Illustration





## HAMILTONIAN PATH Is NP-Complete[a]

**Theorem 19** *Given an* undirected *graph, the question whether it has a Hamiltonian path is NP-complete.*

[a]Karp (1972).

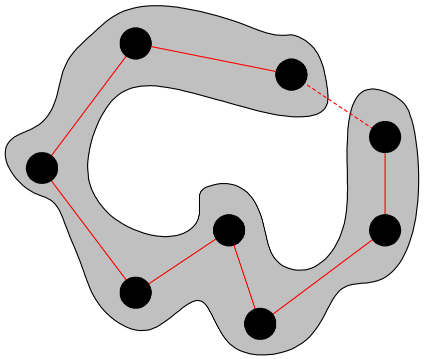


## TSP (D) Is NP-Complete

**Corollary 20** TSP (D) *is NP-complete.*

- Consider a graph $G$ with $n$ nodes.
- Define $d_{ij} = 1$ if $[\,i, j\,] \in G$ and $d_{ij} = 2$ if $[\,i, j\,] \notin G$.
- Set the budget $B = n + 1$.
- If $G$ has no Hamiltonian paths, then every tour on the new graph must contain at least two edges with weight 2.
- The total cost is then at least $(n - 2) + 2 \cdot 2 = n + 2$.
- There is a tour of length $B$ or less if and only if $G$ has a Hamiltonian path.

## Hamiltonian Path and TSP Tour





## Graph Coloring

- $k$-COLORING asks if the nodes of a graph can be colored with $\leq k$ colors such that no two adjacent nodes have the same color.
- 2-COLORING is in P (why?).
- But 3-COLORING is NP-complete (see next page).
- $k$-COLORING is NP-complete for $k \geq 3$ (why?).



## 3-COLORING Is NP-Complete[a]

- We will reduce NAESAT to 3-COLORING.
- We are given a set of clauses $C_1, C_2, \ldots, C_m$ each with 3 literals.
- The boolean variables are $x_1, x_2, \ldots, x_n$.
- We shall construct a graph $G$ such that it can be colored with colors $\{0, 1, 2\}$ if and only if all the clauses can be NAE-satisfied.

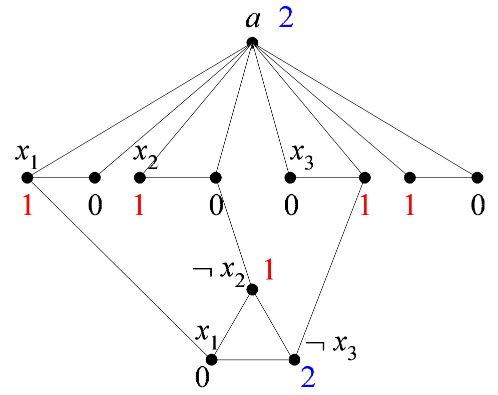
[a] Karp (1972).



## The Proof (continued)

- Every variable $x_i$ is involved in a triangle $[\,a, x_i, \neg x_i\,]$ with a common node $a$.
- Each clause $C_i = (c_{i1} \lor c_{i2} \lor c_{i3})$ is also represented by a triangle
$$[\,c_{i1}, c_{i2}, c_{i3}\,].$$
  - Node $c_{ij}$ with the same label as one in some triangle $[\,a, x_k, \neg x_k\,]$ represent *distinct* nodes.
- There is an edge between $c_{ij}$ and the node that represents the $j$th literal of $C_i$.

## Construction for $\cdots \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge \cdots$






## The Proof (continued)

Suppose the graph is 3-colorable.

- Assume without loss of generality that node $a$ takes the color 2.
- A triangle must use up all 3 colors.
- As a result, one of $x_i$ and $\neg x_i$ must take the color 0 and the other 1.



## The Proof (continued)

- Treat 1 as `true` and 0 as `false`.[a]
  - We were dealing only with those triangles with the $a$ node, not the clause triangles.
- The resulting truth assignment is clearly contradiction free.
- As each clause triangle contains one color 1 and one color 0, the clauses are NAE-satisfied.

[a] The opposite also works.



## The Proof (continued)

Suppose the clauses are NAE-satisfiable.

- Color node $a$ with color 2.
- Color the nodes representing literals by their truth values (color 0 for `false` and color 1 for `true`).
  - We were dealing only with those triangles with the $a$ node, not the clause triangles.

## The Proof (concluded)

- For each clause triangle:
  - Pick any two literals with opposite truth values.
  - Color the corresponding nodes with 0 if the literal is `true` and 1 if it is `false`.
  - Color the remaining node with color 2.
- The coloring is legitimate.
  - If literal $w$ of a clause triangle has color 2, then its color will never be an issue.
  - If literal $w$ of a clause triangle has color 1, then it must be connected up to literal $w$ with color 0.
  - If literal $w$ of a clause triangle has color 0, then it must be connected up to literal $w$ with color 1.



## TRIPARTITE MATCHING

- We are given three sets $B$, $G$, and $H$, each containing $n$ elements.
- Let $T \subseteq B \times G \times H$ be a ternary relation.
- TRIPARTITE MATCHING asks if there is a set of $n$ triples in $T$, none of which has a component in common.
  - Each element in $B$ is matched to a different element in $G$ and different element in $H$.

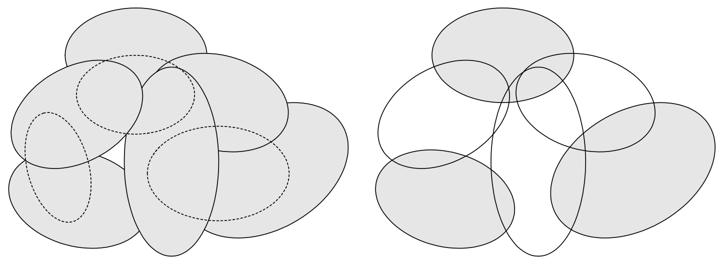
**Theorem 21 (Karp (1972))** TRIPARTITE MATCHING *is NP-complete.*



## Related Problems

- We are given a family $F = \{S_1, S_2, \ldots, S_n\}$ of subsets of a finite set $U$ and a budget $B$.
- SET COVERING asks if there exists a set of $B$ sets in $F$ whose union is $U$.
- SET PACKING asks if there are $B$ disjoint sets in $F$.
- Assume $|U| = 3m$ for some $m \in \mathbb{N}$ and $|S_i| = 3$ for all $i$.
- EXACT COVER BY 3-SETS asks if there are $m$ sets in $F$ that are disjoint and have $U$ as their union.





SET COVERING SET PACKING

## Related Problems (concluded)

**Corollary 22** SET COVERING, SET PACKING, *and* EXACT COVER BY 3-SETS *are all NP-complete.*



## The KNAPSACK Problem

- There is a set of $n$ items.
- Item $i$ has value $v_i \in \mathbb{Z}^+$ and weight $w_i \in \mathbb{Z}^+$.
- We are given $K \in \mathbb{Z}^+$ and $W \in \mathbb{Z}^+$.
- KNAPSACK asks if there exists a subset $S \subseteq \{1, 2, \ldots, n\}$ such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i \geq K$.
  - We want to achieve the maximum satisfaction within the budget.



## KNAPSACK Is NP-Complete

- KNAPSACK $\in$ NP: Guess an $S$ and verify the constraints.
- We assume $v_i = w_i$ for all $i$ and $K = W$.
- KNAPSACK now asks if a subset of $\{w_1, w_2, \ldots, w_n\}$ adds up to exactly $K$.
  - Picture yourself as a radio DJ.
  - Or a person trying to control the calories intake.
- We shall reduce EXACT COVER BY 3-SETS to KNAPSACK.



## The Proof (continued)

- We are given a family $F = \{S_1, S_2, \ldots, S_n\}$ of size-3 subsets of $U = \{1, 2, \ldots, 3m\}$.
- EXACT COVER BY 3-SETS asks if there are $m$ disjoint sets in $F$ that cover the set $U$.
- Think of a set as a bit vector in $\{0, 1\}^{3m}$.
  - `001100010` means the set $\{3, 4, 8\}$, and `110010000` means the set $\{1, 2, 5\}$.
- Our goal is $\overbrace{11 \cdots 1}^{3m}$.

## The Proof (continued)

- A bit vector can also be considered as a binary *number*.
- Set union resembles addition.
  - $\texttt{001100010} + \texttt{110010000} = \texttt{111110010}$, which denotes the set $\{1, 2, 3, 4, 5, 8\}$, as desired.
- Trouble occurs when there is *carry*.
  - $\texttt{001100010} + \texttt{001110000} = \texttt{010010010}$, which denotes the set $\{2, 5, 8\}$, not the desired $\{3, 4, 5, 8\}$.



## The Proof (continued)

- Carry may also lead to a situation where we obtain our solution $\texttt{11}\cdots\texttt{1}$ with more than $m$ sets in $F$.
  - $\texttt{001100010} + \texttt{001110000} + \texttt{101100000} + \texttt{000001101} = \texttt{111111111}$.
  - But this "solution" $\{1, 3, 4, 5, 6, 7, 8, 9\}$ does not correspond to an exact cover.
  - And it uses 4 sets instead of the required 3.[a]
- To fix this problem, we enlarge the base just enough so that there are no carries.
- Because there are $n$ vectors in total, we change the base from 2 to $n + 1$.

[a] Thanks to a lively class discussion on November 20, 2002.



## The Proof (continued)

- Set $v_i$ to be the $(n+1)$-ary number corresponding to the bit vector encoding $S_i$.
- Now in base $n + 1$, if there is a set $S$ such that $\sum_{v_i \in S} v_i = \overbrace{11 \cdots 1}^{3m}$, then every bit position must be contributed by exactly one $v_i$ and $|S| = m$.
- Finally, set
$$K = \sum_{j=0}^{3m-1} (n+1)^j = \overbrace{11 \cdots 1}^{3m} \quad (\text{base } n + 1).$$



## The Proof (continued)

- Suppose $F$ admits an exact cover, say $\{S_1, S_2, \ldots, S_m\}$.
- Then picking $S = \{v_1, v_2, \ldots, v_m\}$ clearly results in
$$v_1 + v_2 + \cdots + v_m = \overbrace{11 \cdots 1}^{3m}.$$
  - It is important to note that the meaning of addition $(+)$ is independent of the base.[a]
  - It is just regular addition.

[a] Contributed by Mr. Kuan-Yu Chen ($\texttt{R92922047}$) on November 3, 2004.

## The Proof (concluded)

- On the other hand, suppose there exists an $S$ such that $\sum_{v_i \in S} v_i = \overbrace{11 \cdots 1}^{3m}$ in base $n+1$.
- The no-carry property implies that $|S| = m$ and $\{S_i : v_i \in S\}$ is an exact cover.




## BIN PACKINGS

- We are given $N$ positive integers $a_1, a_2, \ldots, a_N$, an integer $C$ (the capacity), and an integer $B$ (the number of bins).
- BIN PACKING asks if these numbers can be partitioned into $B$ subsets, each of which has total sum at most $C$.
- Think of packing bags at the check-out counter.

**Theorem 23** BIN PACKING *is NP-complete.*




## INTEGER PROGRAMMING

- INTEGER PROGRAMMING asks whether a system of linear inequalities with integer coefficients has an integer solution.
  - LINEAR PROGRAMMING asks whether a system of linear inequalities with integer coefficients has a *rational* solution.




## INTEGER PROGRAMMING Is NP-Complete[a]

- SET COVERING can be expressed by the inequalities $Ax \geq \vec{1}$, $\sum_{i=1}^{n} x_i \leq B$, $0 \leq x_i \leq 1$, where
  - $x_i$ is one if and only if $S_i$ is in the cover.
  - $A$ is the matrix whose columns are the bit vectors of the sets $S_1, S_2, \ldots$.
  - $\vec{1}$ is the vector of 1s.
- This shows INTEGER PROGRAMMING is NP-hard.
- Many NP-complete problems can be expressed as an INTEGER PROGRAMMING problem.

[a] Papadimitriou (1981).