Reductions and Completeness

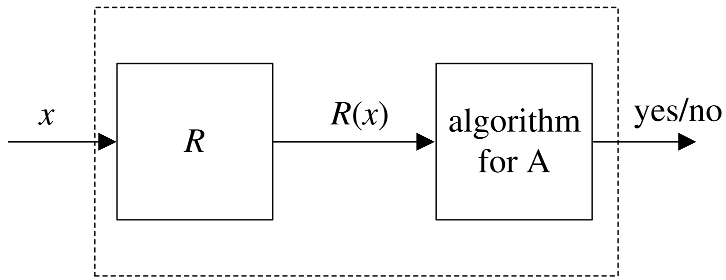
Degrees of Difficulty

- When is a problem more difficult than another?
- B **reduces to** A if there is a transformation R which for every input x of B yields an equivalent input $R(x)$ of A.
  - The answer to x for B is the same as the answer to $R(x)$ for A.
  - There must be restrictions on the complexity of computing R.
  - Otherwise, $R(x)$ might as well solve B.

Degrees of Difficulty (concluded)

- Problem A is at least as hard as problem B if B reduces to A.
- This makes intuitive sense: If A is able to solve your problem B, then A must be at least as powerful.

Reduction



Solving problem B by calling the algorithm for problem *once* and *without* further processing its answer.

Comments[a]

- Suppose B reduces to A via a transformation R.
- The input x is an instance of B.
- The output $R(x)$ is an instance of A.
- $R(x)$ may not span all possible instances of A.
- So some instances of A may never appear in the reduction.

[a]Contributed by Mr. Ming-Feng Tsai (D92922003) on October 29, 2003.

Reduction between Languages

- Language L_1 is **reducible to** L_2 if there is a function R computable by a deterministic TM in polynomial time.
- Furthermore, for all inputs x, $x \in L_1$ if and only if $R(x) \in L_2$.
- R is said to be a **reduction** from L_1 to L_2.
- If R is a reduction from L_1 to L_2, then $R(x) \in L_2$ is a legitimate algorithm for $x \in L_1$.

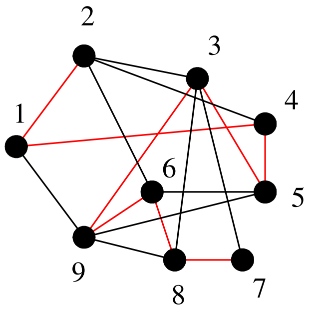
HAMILTONIAN PATH

- A **Hamiltonian path** of a graph is a path that visits every node of the graph exactly once.
- Suppose graph G has n nodes: $1, 2, \ldots, n$.
- A Hamiltonian path can be expressed as a permutation π of $\{ 1, 2, \ldots, n \}$ such that
  - $\pi(i) = j$ means the ith position is occupied by node j.
  - $(\pi(i), \pi(i+1)) \in G$ for $i = 1, 2, \ldots, n-1$.
- HAMILTONIAN PATH asks if a graph has a Hamiltonian path.

Reduction of HAMILTONIAN PATH to SAT

- Given a graph G, we shall construct a CNF $R(G)$ such that $R(G)$ is satisfiable if and only if G has a Hamiltonian path.
- $R(G)$ has n^2 boolean variables x_{ij}, $1 \leq i, j \leq n$.
- x_{ij} means

  the ith position in the Hamiltonian path is occupied by node j.



$x_{12} = x_{21} = x_{34} = x_{45} = x_{53} = x_{69} = x_{76} = x_{88} = x_{97} = 1.$

The Clauses of $R(G)$ and Their Intended Meanings

1. Each node j must appear in the path.
   - $x_{1j} \vee x_{2j} \vee \cdots \vee x_{nj}$ for each j.
2. No node j appears twice in the path.
   - $\neg x_{ij} \vee \neg x_{kj}$ for all i, j, k with $i \neq k$.
3. Every position i on the path must be occupied.
   - $x_{i1} \vee x_{i2} \vee \cdots \vee x_{in}$ for each i.
4. No two nodes j and k occupy the same position in the path.
   - $\neg x_{ij} \vee \neg x_{ik}$ for all i, j, k with $j \neq k$.
5. Nonadjacent nodes i and j cannot be adjacent in the path.
   - $\neg x_{ki} \vee \neg x_{k+1,j}$ for all $(i, j) \notin G$ and $k = 1, 2, \ldots, n - 1$.

The Proof

- $R(G)$ contains $O(n^3)$ clauses.
- $R(G)$ can be computed efficiently (simple exercise).
- Suppose $T \models R(G)$.
- From Clauses of 1 and 2, for each node j there is a unique position i such that $T \models x_{ij}$.
- From Clauses of 3 and 4, for each position i there is a unique node j such that $T \models x_{ij}$.
- So there is a permutation π of the nodes such that $\pi(i) = j$ if and only if $T \models x_{ij}$.

The Proof (concluded)

- Clauses of 5 furthermore guarantees that $(\pi(1), \pi(2), \ldots, \pi(n))$ is a Hamiltonian path.
- Conversely, suppose G has a Hamiltonian path

$$(\pi(1), \pi(2), \ldots, \pi(n)),$$

where π is a permutation.

- Clearly, the truth assignment

$$T(x_{ij}) = \texttt{true} \text{ if and only if } \pi(i) = j$$

satisfies all clauses of $R(G)$.

Reduction of REACHABILITY to CIRCUIT VALUE

- Note that both problems are in P.
- Given a graph $G = (V, E)$, we shall construct a *variable-free* circuit $R(G)$.
- The output of $R(G)$ is true if and only if there is a path from node 1 to node n in G.
- Idea: the Floyd-Warshall algorithm.

The Gates

- The gates are
  - g_{ijk} with $1 \leq i, j \leq n$ and $0 \leq k \leq n$.
  - h_{ijk} with $1 \leq i, j, k \leq n$.
- g_{ijk}: There is a path from node i to node j without passing through a node bigger than k.
- h_{ijk}: There is a path from node i to node j passing through k but not any node bigger than k.
- Input gate $g_{ij0} = \texttt{true}$ if and only if $i = j$ or $(i, j) \in E$.

The Construction

- h_{ijk} is an AND gate with predecessors $g_{i,k,k-1}$ and $g_{k,j,k-1}$, where $k = 1, 2, \ldots, n$.
- g_{ijk} is an OR gate with predecessors $g_{i,j,k-1}$ and $h_{i,j,k}$, where $k = 1, 2, \ldots, n$.
- g_{1nn} is the output gate.
- Interestingly, $R(G)$ uses no $\neg$ gates: It is a **monotone circuit**.

Reduction of CIRCUIT SAT to SAT

- Given a circuit C, we shall construct a boolean expression $R(C)$ such that $R(C)$ is satisfiable if and only if C is satisfiable.
  - $R(C)$ will turn out to be a CNF.
- The variables of $R(C)$ are those of C plus g for each gate g of C.
- Each gate of C will be turned into equivalent clauses of $R(C)$.
- Recall that clauses are $\wedge$-ed together.

The Clauses of $R(C)$

g **is a variable gate** x**:** Add clauses $(\neg g \lor x)$ and $(g \lor \neg x)$.

- Meaning: $g \Leftrightarrow x$.

g **is a `true` gate:** Add clause (g).

- Meaning: g must be true to make $R(C)$ true.

g **is a `false` gate:** Add clause $(\neg g)$.

- Meaning: g must be false to make $R(C)$ true.

g **is a $\neg$ gate with predecessor gate** h**:** Add clauses $(\neg g \lor \neg h)$ and $(g \lor h)$.

- Meaning: $g \Leftrightarrow \neg h$.

The Clauses of $R(C)$ (concluded)

g **is a $\lor$ gate with predecessor gates** h **and** h'**:** Add clauses $(\neg h \lor g)$, $(\neg h' \lor g)$, and $(h \lor h' \lor \neg g)$.

- Meaning: $g \Leftrightarrow (h \lor h')$.

g **is a $\land$ gate with predecessor gates** h **and** h'**:** Add clauses $(\neg g \lor h)$, $(\neg g \lor h')$, and $(\neg h \lor \neg h' \lor g)$.

- Meaning: $g \Leftrightarrow (h \land h')$.

g **is the output gate:** Add clause (g).

- Meaning: g must be true to make $R(C)$ true.

Composition of Reductions

Proposition 10 *If R_{12} is a reduction from L_1 to L_2 and R_{23} is a reduction from L_2 to L_3, then the composition $R_{12} \circ R_{23}$ is a reduction from L_1 to L_3.*

- Clearly $x \in L_1$ if and only if $R_{23}(R_{12}(x)) \in L_3$.
- It is also clear that $R_{12} \circ R_{23}$ can be computed in polynomial time.

Completeness[a]

- As reducibility is transitive, problems can be ordered with respect to their difficulty.
- Is there a *maximal* element?
- It is not altogether obvious that there should be a maximal element.
- Many infinite structures (such as integers and reals) do not have maximal elements.
- Hence it may surprise you that most of the complexity classes that we have seen so far have maximal elements.

[a]Cook (1971).

Completeness (concluded)

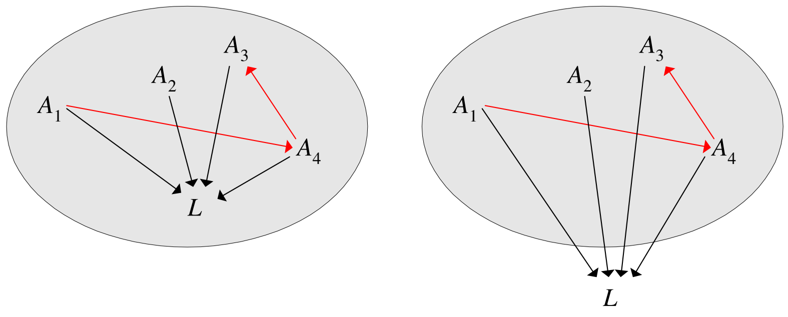
- Let $\mathcal{C}$ be a complexity class and $L \in \mathcal{C}$.
- L is **$\mathcal{C}$-complete** if every $L' \in \mathcal{C}$ can be reduced to L.
  - Most complexity classes we have seen so far have complete problems!
- Complete problems capture the difficulty of a class because they are the hardest.

Hardness

- Let $\mathcal{C}$ be a complexity class.
- L is **$\mathcal{C}$-hard** if every $L' \in \mathcal{C}$ can be reduced to L.
- It is not required that $L \in \mathcal{C}$.
- If L is $\mathcal{C}$-hard, then by definition, every $\mathcal{C}$-complete problem can be reduced to L.[a]

[a]Contributed by Mr. Ming-Feng Tsai (`D92922003`) on October 15, 2003.

Illustration of Completeness and Hardness



Closedness under Reduction

- A class $\mathcal{C}$ is **closed under reductions** if whenever L is reducible to L' and $L' \in \mathcal{C}$, then $L \in \mathcal{C}$.
- P, NP, and EXP are all closed under reductions.

Complete Problems and Complexity Classes

Proposition 11 *Let $\mathcal{C}'$ and $\mathcal{C}$ be two complexity classes such that $\mathcal{C}' \subseteq \mathcal{C}$. Assume $\mathcal{C}'$ is closed under reductions and L is a complete problem for $\mathcal{C}$. Then $\mathcal{C} = \mathcal{C}'$ if and only if $L \in \mathcal{C}'$.*

- Suppose $L \in \mathcal{C}'$ first.
- Every language $A \in \mathcal{C}$ reduces to $L \in \mathcal{C}'$.
- Because $\mathcal{C}'$ is closed under reductions, $A \in \mathcal{C}'$.
- Hence $\mathcal{C} \subseteq \mathcal{C}'$.
- As $\mathcal{C}' \subseteq \mathcal{C}$, we conclude that $\mathcal{C} = \mathcal{C}'$.

The Proof (concluded)

- On the other hand, suppose $\mathcal{C} = \mathcal{C}'$.
- As L is $\mathcal{C}$-complete, $L \in \mathcal{C}$.
- Thus, trivially, $L \in \mathcal{C}'$.

Two Immediate Corollaries

Proposition 11 implies that

- P = NP if and only if an NP-complete problem in P.
- L = P if and only if a P-complete problem is in L.

Complete Problems and Complexity Classes

Proposition 12 *Let $\mathcal{C}'$ and $\mathcal{C}$ be two complexity classes closed under reductions. If L is complete for both $\mathcal{C}$ and $\mathcal{C}'$, then $\mathcal{C} = \mathcal{C}'$.*

- All languages $\mathcal{L} \in \mathcal{C}$ reduce to $L \in \mathcal{C}'$.
- Since $\mathcal{C}'$ is closed under reductions, $\mathcal{L} \in \mathcal{C}'$.
- Hence $\mathcal{C} \subseteq \mathcal{C}'$.
- The proof for $\mathcal{C}' \subseteq \mathcal{C}$ is symmetric.

Table of Computation

- Let $M = (K, \Sigma, \delta, s)$ be a single-string polynomial-time deterministic TM deciding L.
- Its computation on input x can be thought of as a $|x|^k \times |x|^k$ table, where $|x|^k$ is the time bound (recall that it is an upper bound).
  - It is a sequence of configurations.
- Rows correspond to time steps 0 to $|x|^k - 1$.
- Columns are positions in the string of M.
- The (i, j)th table entry represents the contents of position j of the string *after* i steps of computation.

Some Conventions To Simplify the Table

- M halts after at most $|x|^k - 2$ steps.
  - The string length hence never exceeds $|x|^k$.
- Assume a large enough k to make it true for $|x| \geq 2$.
- Pad the table with $\sqcup$s so that each row has length $|x|^k$.
  - The computation will never reach the right end of the table for lack of time.
- If the cursor scans the jth position at time i when M is at state q and the symbol is σ, then the (i, j)th entry is a *new* symbol σ_q.

Some Conventions To Simplify the Table (continued)

- If q is "yes" or "no," simply use "yes" or "no" instead of σ_q.
- Modify M so that the cursor starts not at $\triangleright$ but at the first symbol of the input.
- The cursor never visits the leftmost $\triangleright$ by telescoping two moves of M each time the cursor is about to move to the leftmost $\triangleright$.
- So the first symbol in every row is a $\triangleright$ and not a $\triangleright_q$.

Some Conventions To Simplify the Table (concluded)

- If M has halted before its time bound of $|x|^k$, so that "yes" or "no" appears at a row before the last, then all subsequent rows will be identical to that row.
- M accepts x if and only if the $(|x|^k - 1, j)$th entry is "yes" for some j.

Comments

- Each row is essentially a configuration.
- If the input $x = 010001$, then the first row is

$$\overbrace{\triangleright 0_s 10001 \sqcup \sqcup \cdots \sqcup}^{|x|^k}$$

- A typical row may be

$$\overbrace{\triangleright 10100_q 01110100 \sqcup \sqcup \cdots \sqcup}^{|x|^k}$$

- The last rows must look like $\overbrace{\triangleright \cdots \text{"yes"} \cdots \sqcup}^{|x|^k}$

A P-Complete Problem

Theorem 13 (Ladner (1975)) CIRCUIT VALUE *is P-complete.*

- It is easy to see that CIRCUIT VALUE $\in$ P.
- For any $L \in$ P, we will construct a reduction R from L to CIRCUIT VALUE.
- Given any input x, $R(x)$ is a variable-free circuit such that $x \in L$ if and only if $R(x)$ evaluates to true.
- Let M decide L in time n^k.
- Let T be the computation table of M on x.

The Proof (continued)

- When $i = 0$, or $j = 0$, or $j = |x|^k - 1$, then the value of T_{ij} is known.
  - The jth symbol of x or $\sqcup$, a $\triangleright$, and a $\sqcup$, respectively.
  - Three out of four of T's borders are known.

$\triangleright$	a	b	c	d	e	f	$\sqcup$
$\triangleright$							$\sqcup$
$\triangleright$							$\sqcup$
$\triangleright$							$\sqcup$
$\triangleright$							$\sqcup$

The Proof (continued)

- Consider *other* entries T_{ij}.
- T_{ij} depends on only $T_{i-1,j-1}$, $T_{i-1,j}$, and $T_{i-1,j+1}$.

$T_{i-1,j-1}$	$T_{i-1,j}$	$T_{i-1,j+1}$
	T_{ij}	

- Let Γ denote the set of all symbols that can appear on the table: $\Gamma = \Sigma \cup \{\sigma_q : \sigma \in \Sigma, q \in K\}$.
- Encode each symbol of Γ as an m-bit number, where

$$m = \lceil \log_2 |\Gamma| \rceil$$

(**state assignment** in circuit design).

The Proof (continued)

- Let binary string $S_{ij1} S_{ij2} \cdots S_{ijm}$ encode T_{ij}.
- We may treat them interchangeably without ambiguity.
- The computation table is now a table of binary entries $S_{ij\ell}$, where

$$0 \le i \le n^k - 1,$$
$$0 \le j \le n^k - 1,$$
$$1 \le \ell \le m.$$

The Proof (continued)

- Each bit $S_{ij\ell}$ depends on only $3m$ other bits:

$T_{i-1,j-1}$:	$S_{i-1,j-1,1}$	$S_{i-1,j-1,2}$	$\cdots$	$S_{i-1,j-1,m}$
$T_{i-1,j}$:	$S_{i-1,j,1}$	$S_{i-1,j,2}$	$\cdots$	$S_{i-1,j,m}$
$T_{i-1,j+1}$:	$S_{i-1,j+1,1}$	$S_{i-1,j+1,2}$	$\cdots$	$S_{i-1,j+1,m}$

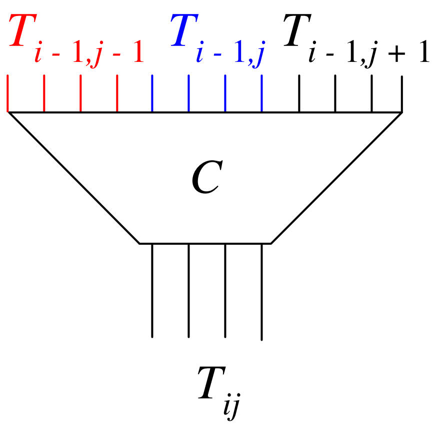
- So there are m boolean functions $F_1, F_2, \ldots, F_m$ with $3m$ inputs each such that for all $i, j > 0$,

$$\begin{aligned} S_{ij\ell} &= F_\ell(S_{i-1,j-1,1}, S_{i-1,j-1,2}, \ldots, S_{i-1,j-1,m}, \\ &\quad S_{i-1,j,1}, S_{i-1,j,2}, \ldots, S_{i-1,j,m}, \\ &\quad S_{i-1,j+1,1}, S_{i-1,j+1,2}, \ldots, S_{i-1,j+1,m}). \end{aligned}$$

The Proof (continued)

- These F_i's depend on only M's specification, not on x.
- Their sizes are fixed.
- These boolean functions can be turned into boolean circuits.
- Compose these m circuits in parallel to obtain circuit C with $3m$-bit inputs and m-bit outputs.
  - Schematically, $C(T_{i-1,j-1}, T_{i-1,j}, T_{i-1,j+1}) = T_{ij}$.
  - C is like an ASIC (application-specific IC) chip.

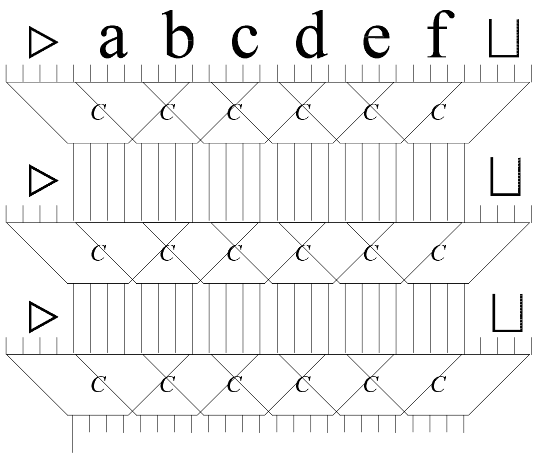
Circuit C



The Proof (concluded)

- A copy of circuit C is placed at each entry of the table.
  - Exceptions are the top row and the two extreme columns.
- $R(x)$ consists of $(|x|^k - 1)(|x|^k - 2)$ copies of circuit C.
- Without loss of generality, assume the output "yes"/"no" (coded as 1/0) appear at position $(|x|^k - 1, 1)$.

The Computation Tableau and $R(x)$



A Corollary

The construction in the above proof shows the following.

Corollary 14 *If $L \in TIME(T(n))$, then a circuit with $O(T^2(n))$ gates can decide if $x \in L$ for $|x| = n$.*

Cook's Theorem: the First NP-Complete Problem

Theorem 15 (Cook (1971)) SAT *is NP-complete.*

- SAT $\in$ NP (p. 49).
- CIRCUIT SAT reduces to SAT (p. 121).
- Now we only need to show that all languages in NP can be reduced to CIRCUIT SAT.

The Proof (continued)

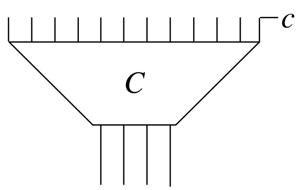
- Let single-string NTM M decide $L \in \mathrm{NP}$ in time n^k.
- Assume M has exactly *two* nondeterministic choices at each step: choices 0 and 1.
- For each input x, we construct circuit $R(x)$ such that $x \in L$ if and only if $R(x)$ is satisfiable.
- A sequence of nondeterministic choices is a bit string

$$B = (c_1, c_2, \ldots, c_{|x|^k-1}) \in \{0,1\}^{|x|^k-1}.$$

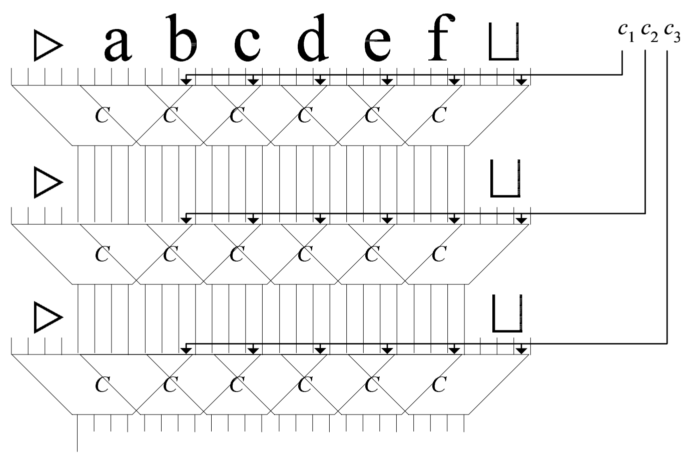
- Once B is fixed, the computation is *deterministic*.

The Proof (continued)

- Each choice of B results in a deterministic polynomial-time computation, hence a table like the one on p. 147.
- Each circuit C at time i has an extra binary input c corresponding to the nondeterministic choice: $C(T_{i-1,j-1}, T_{i-1,j}, T_{i-1,j+1}, c) = T_{ij}$.



The Computation Tableau for NTMs and $R(x)$



The Proof (concluded)

- The overall circuit $R(x)$ (on p. 152) is satisfiable if there is a truth assignment B such that the computation table accepts.
- This happens if and only if M accepts x, i.e., $x \in L$.

NP-Complete Problems

Wir müssen wissen, wir werden wissen.
(We must know, we shall know.)
— David Hilbert (1900)

Two Notions

- Let $R \subseteq \Sigma^* \times \Sigma^*$ be a binary relation on strings.
- R is called **polynomially decidable** if
$$\{x; y : (x, y) \in R\}$$
is in P.
- R is said to be **polynomially balanced** if $(x, y) \in R$ implies $|y| \leq |\, x \,|^k$ for some $k \geq 1$.

3SAT

- k-SAT, where $k \in \mathbb{Z}^+$, is the special case of SAT.
- The formula is in CNF and all clauses have *exactly* k literals (repetition of literals is allowed).
- For example,
$$(x_1 \lor x_2 \lor \neg x_3) \land (x_1 \lor x_1 \lor \neg x_2) \land (x_1 \lor \neg x_2 \lor \neg x_3).$$

3SAT Is NP-Complete

- Recall Cook's Theorem (p. 149) and the reduction of CIRCUIT SAT to SAT (p. 121).
- The resulting CNF has at most 3 literals for each clause.
  - This shows that 3SAT where each clause has at most 3 literals is NP-complete.
- Finally, duplicate one literal once or twice to make it a 3SAT formula.

Another Variant of 3SAT

Proposition 16 3SAT *is NP-complete for expressions in which each variable is restricted to appear at most three times, and each literal at most twice. (*3SAT *here requires only that each clause has* at most *3 literals.)*

- Consider a general 3SAT expression in which x appears k times.
- Replace the first occurrence of x by x_1, the second by x_2, and so on, where $x_1, x_2, \ldots, x_k$ are k *new* variables.

The Proof (concluded)

- Add $(\neg x_1 \lor x_2) \land (\neg x_2 \lor x_3) \land \cdots \land (\neg x_k \lor x_1)$ to the expression.
  - This is logically equivalent to $x_1 \Rightarrow x_2 \Rightarrow \cdots \Rightarrow x_k \Rightarrow x_1$.
  - Note that each clause above has fewer than 3 literals.
- The resulting equivalent expression satisfies the condition for x.

NAESAT

- The NAESAT (for "not-all-equal" SAT) is like 3SAT.
- But we require additionally that there be a satisfying truth assignment under which no clauses have the three literals equal in truth value.
  - Each clause must have one literal assigned true and one literal assigned false.

NAESAT Is NP-Complete[a]

- Recall the reduction of CIRCUIT SAT to SAT on p. 121.
- It produced a CNF ϕ in which each clause has at most 3 literals.
- Add the same variable z to all clauses with fewer than 3 literals to make it a 3SAT formula.
- Goal: The new formula $\phi(z)$ is NAE-satisfiable if and only if the original circuit is satisfiable.

[a]Karp (1972).

The Proof (continued)

- Suppose T NAE-satisfies $\phi(z)$.
  - $\bar{T}$ also NAE-satisfies $\phi(z)$.
  - Under T or $\bar{T}$, variable z takes the value false.
  - This truth assignment must still satisfy all clauses of ϕ.
  - So it satisfies the original circuit.

The Proof (concluded)

- Suppose there is a truth assignment that satisfies the circuit.
  - Then there is a truth assignment T that satisfies every clause of ϕ.
  - Extend T by adding $T(z) = \texttt{false}$ to obtain T'.
  - T' satisfies $\phi(z)$.
  - So in no clauses are all three literals false under T'.
  - Under T', in no clauses are all three literals true.
    - Review the detailed construction on p. 122 and p. 123.

Undirected Graphs

- An **undirected graph** $G = (V, E)$ has a finite set of nodes, V, and a set of *undirected* edges, E.
- It is like a directed graph except that the edges have no directions and there are no self-loops.
- We use $[\, i, j \,]$ to denote the fact that there is an edge between node i and node j.

Independent Sets

- Let $G = (V, E)$ be an undirected graph.
- $I \subseteq V$.
- I is **independent** if whenever $i, j \in I$, there is no edge between i and j.
- The INDEPENDENT SET problem: Given an undirected graph and a goal K, is there an independent set of size K?
  - Many applications.

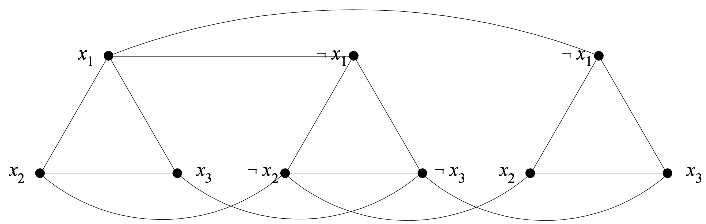
INDEPENDENT SET Is NP-Complete

- This problem is in NP: Guess a set of nodes and verify that it is independent and meets the count.
- If a graph contains a triangle, any independent set can contain at most one node of the triangle.
- We consider graphs whose nodes can be partitioned in m disjoint triangles.
  - If the special case is hard, the original problem must be at least as hard.

Reduction from 3SAT to INDEPENDENT SET

- Let ϕ be an instance of 3SAT with m clauses.
- We will construct graph G (with constraints as said) with $K = m$ such that ϕ is satisfiable if and only if G has an independent set of size K.
- There is a triangle for each clause with the literals as the nodes.
- Add additional edges between x and $\neg x$ for every variable x.

A Sample Construction



$$(x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor x_2 \lor x_3).$$

The Proof (continued)

- Suppose G has an independent set I of size $K = m$.
  - An independent set can contain at most m nodes, one from each triangle.
  - An independent set of size m exists if and only if it contains exactly one node from each triangle.
  - Truth assignment T assigns true to those literals in I.
  - T is consistent because contradictory literals are connected by an edge, hence not both in I.
  - T satisfies ϕ because it has a node from every triangle, thus satisfying every clause.

The Proof (concluded)

- Suppose a satisfying truth assignment T exists for ϕ.
  - Collect one node from each triangle whose literal is true under T.
  - The choice is arbitrary if there is more than one true literal.
  - This set of m nodes must be independent by construction.
    - Literals x and $\neg x$ cannot be both assigned true.