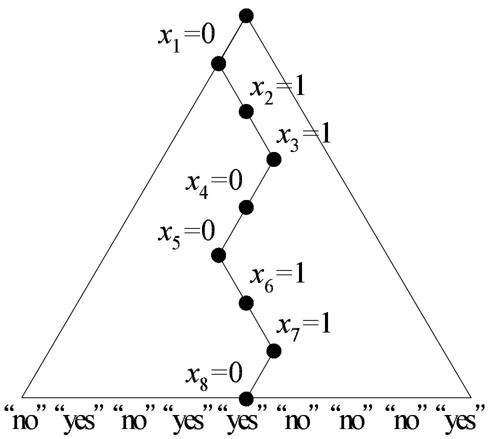
A Nondeterministic Algorithm for Satisfiability

ϕ is a boolean formula with n variables.

for $i = 1, 2, \ldots, n$ **do**
    Guess $x_i \in \{0, 1\}$; {Nondeterministic choice.}
end for
{Verification:}
if $\phi(x_1, x_2, \ldots, x_n) = 1$ **then**
    "yes";
else
    "no";
end if

The Computation Tree for Satisfiability



Analysis

- The algorithm decides language $\{\phi : \phi \text{ is satisfiable}\}$.
  - The computation tree is a complete binary tree of depth n.
  - Every computation path corresponds to a particular truth assignment out of 2^n.
  - ϕ is satisfiable if and only if there is a computation path (truth assignment) that results in "yes."
- General paradigm: Guess a "proof" and then verify it.

The Traveling Salesman Problem

- We are given n cities $1, 2, \ldots, n$ and integer distances d_{ij} between any two cities i and j.
- Assume $d_{ij} = d_{ji}$ for convenience.
- The **traveling salesman problem** (TSP) asks for the total distance of the shortest tour of the cities.
- The decision version TSP (D) asks if there is a tour with a total distance at most B, where B is an input.
- Both problems are extremely important but equally hard (p. 308 and p. 370).

A Nondeterministic Algorithm for TSP (D)

for $i = 1, 2, \ldots, n$ **do**
  Guess $x_i \in \{1, 2, \ldots, n\}$; {The ith city.}
end for
$x_{n+1} := x_1$;
{Verification stage:}
if $x_1, x_2, \ldots, x_n$ are distinct and $\sum_{i=1}^{n} d_{x_i, x_{i+1}} \leq B$ **then**
  "yes";
else
  "no";
end if

(The degree of nondeterminism is n.)

Time Complexity under Nondeterminism

- Nondeterministic machine N decides L **in time** $f(n)$, where $f : \mathbb{N} \to \mathbb{N}$, if
  - N decides L, and
  - for any $x \in \Sigma^*$, N does not have a computation path longer than $f(|x|)$.
- We charge only the "depth" of the computation tree.

Time Complexity Classes under Nondeterminism

- NTIME$(f(n))$ is the set of languages decided by NTMs within time $f(n)$.
- NTIME$(f(n))$ is a complexity class.

NP

- Define
$$\text{NP} = \bigcup_{k>0} \text{NTIME}(n^k).$$
- Clearly P $\subseteq$ NP.
- Think of NP as efficiently *verifiable* problems.
  - Boolean satisfiability (SAT).
  - TSP (D).
  - Hamiltonian path.
  - Graph colorability.
- The most important open problem in computer science is whether P = NP.

Simulating Nondeterministic TMs

Theorem 5 *Suppose language L is decided by an NTM N in time $f(n)$. Then it is decided by a 3-string deterministic TM M in time $O(c^{f(n)})$, where $c > 1$ is some constant depending on N.*

- On input x, M goes down every computation path of N using *depth-first* search (but M does *not* know $f(n)$).
- If some path leads to "yes," then M enters the "yes" state.
- If none of the paths leads to "yes," then M enters the "no" state.

NTIME vs. TIME

Corollary 6 NTIME$(f(n))) \subseteq \bigcup_{c>1}$ TIME$(c^{f(n)})$.

- Does converting an NTM into a TM require exploring all the computation paths of the NTM as done in Theorem 5?
- This is the most important question in theory with practical implications.

Graph Reachability

- Let $G(V, E)$ be a directed graph (digraph).
- REACHABILITY asks if, given nodes a and b, does G contain a path from a to b?
- Can be easily solved in polynomial time by breadth-first search.
- How about the nondeterministic space complexity?

The First Try in NSPACE$(n \log n)$

$x_1 := a$; {Assume $a \neq b$.}
for $i = 2, 3, \ldots, n$ **do**
  Guess $x_i \in \{v_1, v_2, \ldots, v_n\}$; {The ith node.}
end for
for $i = 2, 3, \ldots, n$ **do**
  if $(x_{i-1}, x_i) \notin E$ **then**
    "no";
  end if
  if $x_i = b$ **then**
    "yes";
  end if
end for
"no";

In Fact REACHABILITY $\in$ NSPACE$(\log n)$

$x := a$;
for $i = 2, 3, \ldots, n$ **do**
  Guess $y \in \{2, 3, \ldots, n\}$; {The next node.}
  if $(x, y) \notin E$ **then**
    "no";
  end if
  if $y = b$ **then**
    "yes";
  end if
  $x := y$;
end for
"no";

Space Analysis

- Variables i, x, and y each require $O(\log n)$ bits.
- Testing $(x, y) \in E$ is accomplished by consulting the input string with counters of $O(\log n)$ bits long.
- Hence

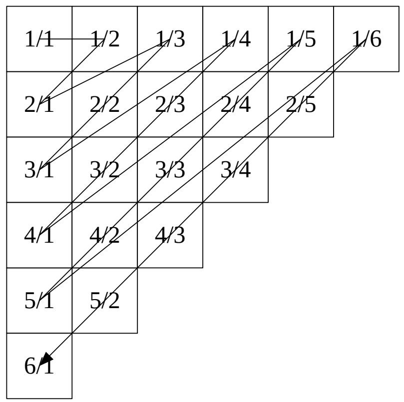
$$\text{REACHABILITY} \in \text{NSPACE}(\log n).$$

  - REACHABILITY with more than one terminal node also has the same complexity.
- REACHABILITY $\in$ P (p. 175).

Infinite Sets

- A set is **countable** if it is finite or if it can be put in one-one correspondence with $\mathbb{N}$, the set of natural numbers.
  - Set of integers $\mathbb{Z}$.
    * $0 \leftrightarrow 0, 1 \leftrightarrow 1, 2 \leftrightarrow 3, 3 \leftrightarrow 5, \ldots, -1 \leftrightarrow 2, -2 \leftrightarrow 4, -3 \leftrightarrow 6, \ldots$.
  - Set of positive integers $\mathbb{Z}^+$: $i - 1 \leftrightarrow i$.
  - Set of odd integers: $(i - 1)/2 \leftrightarrow i$.
  - Set of rational numbers: See next page.
  - Set of squared integers: $i \leftrightarrow \sqrt{i}$.

Rational Numbers Are Countable



Cardinality

- For any set A, define $|A|$ as A's **cardinality** (size).
- Two sets are said to have the same cardinality (written as $|A| = |B|$ or $A \sim B$) if there exists a one-to-one correspondence between their elements.
- 2^A denotes set A's **power set**, that is $\{B : B \subseteq A\}$.
  - If $|A| = k$, then $|2^A| = 2^k$.
  - So $|A| < |2^A|$ when A is finite.

Cardinality (concluded)

- $|A| \leq |B|$ if there is a one-to-one correspondence between A and one of B's subsets.
- $|A| < |B|$ if $|A| \leq |B|$ but $|A| \neq |B|$.
- If $A \subseteq B$, then $|A| \leq |B|$.
- But if $A \subsetneq B$, then $|A| < |B|$?

Cardinality and Infinite Sets

- If A and B are infinite sets, it is possible that $A \subsetneq B$ yet $|A| = |B|$.
  - The set of integers *properly* contains the set of odd integers.
  - But the set of integers has the same cardinality as the set of odd integers (p. 94).
- A lot of "paradoxes."

Hilbert's[a] Paradox of the Grand Hotel

- For a hotel with a finite number of rooms with all the rooms occupied, a new guest will be turned away.
- Now let us imagine a hotel with an infinite number of rooms, and all the rooms are occupied.
- A new guest comes and asks for a room.
- "But of course!" exclaims the proprietor, and he moves the person previously occupying Room 1 into Room 2, the person from Room 2 into Room 3, and so on ....
- The new customer occupies Room 1.

[a]David Hilbert (1862–1943).

Hilbert's Paradox of the Grand Hotel (concluded)

- Let us imagine now a hotel with an infinite number of rooms, all taken up, and an infinite number of new guests who come in and ask for rooms.
- "Certainly, gentlemen," says the proprietor, "just wait a minute."
- He moves the occupant of Room 1 into Room 2, the occupant of Room 2 into Room 4, and so on.
- Now all odd-numbered rooms become free and the infinity of new guests can be accommodated in them.
- "There are many rooms in my Father's house, and I am going to prepare a place for you." (*John* 14:3)

Galileo's[a] Paradox (1638)

- The squares of the positive integers can be placed in one-to-one correspondence with all the positive integers.
- This is contrary to the axiom of Euclid that the whole is greater than any of its proper parts.
- Resolution of paradoxes: Pick the notion that results in "better" mathematics.
- The difference between a mathematical paradox and a contradiction is often a matter of opinion.

[a]Galileo (1564–1642).

Cantor's[a] Theorem

Theorem 7 *The set of all subsets of* $\mathbb{N}$ $(2^{\mathbb{N}})$ *is infinite and not countable.*

- Suppose it is countable with $f : \mathbb{N} \to 2^{\mathbb{N}}$ being a bijection.
- Consider the set $B = \{k \in \mathbb{N} : k \not\in f(k)\} \subseteq \mathbb{N}$.
- Suppose $B = f(n)$ for some $n \in \mathbb{N}$.

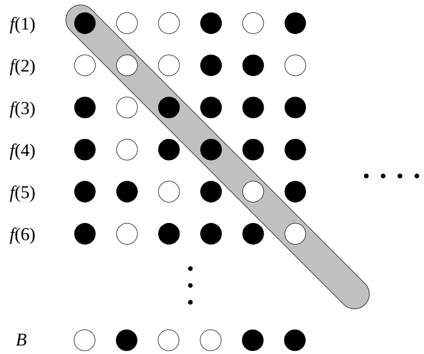
[a]Georg Cantor (1845–1918).

The Proof (concluded)

- If $n \in f(n)$, then $n \in B$, but then $n \not\in B$ by B's definition.[a]
- If $n \not\in f(n)$, then $n \not\in B$, but then $n \in B$ by B's definition.
- Hence $B \neq f(n)$ for any n.
- f is not a bijection, a contradiction.

[a]If B is empty, skip this part. Thanks to a lively class discussion on October 1, 2003.

Cantor's Diagonalization Argument Illustrated



How about ... ?[a]

- Consider this subset of $2^{\mathbb{N}}$:

$$2^{\mathbb{N}}_{=k} \equiv \{\, x : x \subseteq \mathbb{N}, |\, x \,| = k \,\}.$$

- Is it still uncountable?
- No.
  - $|\, 2^{\mathbb{N}}_{=1} \,| = |\, \mathbb{N} \,|$.
  - $|\, 2^{\mathbb{N}}_{=2} \,| = |\, \mathbb{Q} \,|$.

[a] Thanks to a lively class discussion on October 1, 2003.

A Corollary of Cantor's Theorem

Corollary 8 *For any set T, finite or infinite,*

$$|\, T \,| < |\, 2^T \,|.$$

- The inequality holds in the finite A case.
- Assume A is infinite now.
- $|T| \leq |2^T|$: Consider $f(x) = \{x\}$.
- The strict inequality uses the same argument as Cantor's theorem.

A Second Corollary of Cantor's Theorem

Corollary 9 *The set of all functions on $\mathbb{N}$ is not countable.*

- Every function $f : \mathbb{N} \to \{0, 1\}$ determines a set

$$\{n : f(n) = 1\} \subseteq \mathbb{N}.$$

- And vice versa.
- So the set of functions from $\mathbb{N}$ to $\{0, 1\}$ has cardinality $|\, 2^{\mathbb{N}} \,|$.
- Corollary 8 (p. 106) then implies the claim.

Existence of Uncomputable Problems

- Every program is a finite sequence of 0s and 1s, thus a nonnegative integer.
- Hence every program corresponds to some integer.
- The set of programs is countable.
- A function is a mapping from integers to integers.
- The set of functions is not countable by Corollary 9 (p. 107).
- So there must exist functions for which there are no programs.

Universal Turing Machine[a]

- A **universal Turing machine** U interprets the input as the *description* of a TM M concatenated with the *description* of an input to that machine, x.
  - Both M and x are over the alphabet of U.
- U simulates M on x so that

$$U(M;x) = M(x).$$

- U is like a modern computer, which executes any valid machine code, or a Java Virtual machine, which executes any valid bytecode.

[a] Turing (1936).

The Halting Problem

- **Undecidable problems** are problems that have no algorithms or languages that are not recursive.
- We knew undecidable problems exist (p. 108).
- We now define a concrete undecidable problem, the **halting problem**:

$$H = \{M;x : M(x) \neq \nearrow\}.$$

  - Does M halt on input x?

H Is Recursively Enumerable

- Use the universal TM U to simulate M on x.
- When M is about to halt, U enters a "yes" state.
- If $M(x)$ diverges, so does U.
- This TM accepts H.
- Membership of x in any recursively enumerative language accepted by M can be answered by asking

$$M;x \in H?$$

H Is Not Recursive

- Suppose there is a TM M_H that *decides* H.
- Consider the program $D(M)$ that calls M_H:

if $M_H(M; M) =$ "yes" **then**
 $\nearrow$; {Writing an infinite loop is easy, right?}
else
 "yes";
end if

- Consider $D(D)$:
  - $D(D) = \nearrow \Rightarrow M_H(D; D) =$ "yes" $\Rightarrow D; D \in H \Rightarrow D(D) \neq \nearrow$, a contradiction.
  - $D(D) =$ "yes" $\Rightarrow M_H(D; D) =$ "no" $\Rightarrow D; D \notin H \Rightarrow D(D) = \nearrow$, a contradiction.

Comments

- In general, we cannot tell if a running program will ever halt.
- Two levels of interpretations of M:
  - A sequence of 0s and 1s (data).
  - An encoding of instructions (programs).
- There are no paradoxes.
  - Concepts should be familiar to computer scientists.
  - Supply a C compiler to a C compiler, a Lisp interpreter to a Lisp interpreter, etc.

Self-Loop Paradoxes

Cantor's Paradox (1899): Let T be the set of all sets.

- Then $2^T \subseteq T$, but we know $|2^T| > |T|$!

Russell's[a] Paradox (1901): Consider $R = \{A : A \notin A\}$.

- If $R \in R$, then $R \notin R$ by definition.
- If $R \notin R$, then $R \in R$ also by definition.

Eubulides: The Cretan says, "All Cretans are liars."

Sharon Stone in *The Specialist* (1994): "I'm not a woman you can trust."

Axiomatic Set Theory

- Russell's paradox initiated the effort to axiomatize set theory in 1908–1929.
- The standard theory is the Zermelo-Fraenkel-Skolem (ZFS) system.[a]
- In ZFS, the Axiom of Foundation says that any descending membership chain is finite.
- Then $x \notin x$ for any set x.
  - Otherwise, $x \in x \in x \in \cdots$, a contradiction
- Hence Russell's paradox is avoided.

[a]Ernst Friedrich Ferdinand Zermelo (1871–1953); Adolf Abraham Halevi Fraenkel (1891–1965); Albert Thoralf Skolem (1887–1963).

More Undecidability

- $\{M : M \text{ halts on all inputs}\}$.
  - Given $M; x$, we construct the following machine:
    * $M_x(y)$: if $y = x$ then $M(x)$ else halt.
  - M_x halts on all inputs if and only if M halts on x.
  - So if the said language were recursive, H would be recursive, a contradiction.
  - This technique is called **reduction**.

More Undecidability (concluded)

- $\{M; x : \text{there is a } y \text{ such that } M(x) = y\}$.
- $\{M; x : \text{the computation } M \text{ on input } x \text{ uses all states of } M\}$.
- $\{M; x; y : M(x) = y\}$.

Reductions in Proving Undecidability

- Suppose we are asked to prove L is undecidable.
- Language H is known to be undecidable.
- We try to find a computable transformation (or reduction) R such that

$$R(x) \in L \text{ if and only if } x \in H.$$

- This suffices to prove that L is undecidable.

Complements of Recursive Languages

Lemma 10 *If L is recursive, then so is $\bar{L}$.*

- Let L be decided by M (which is deterministic).
- Swap the "yes" state and the "no" state of M.
- The new machine decides $\bar{L}$.

Recursive and Recursively Enumerable Languages

Lemma 11 *L is recursive if and only if both L and $\bar{L}$ are recursively enumerable.*

- Suppose both L and $\bar{L}$ are recursively enumerable, accepted by M and $\bar{M}$, respectively.
- Simulate M and $\bar{M}$ in an *interleaved* fashion.
- If M accepts, then $x \in L$ and M' halts on state "yes."
- If $\bar{M}$ accepts, then $x \notin L$ and M' halts on state "no."

A Very Useful Corollary and Its Consequences

Corollary 12 *L is recursively enumerable but not recursive, then $\bar{L}$ is not recursively enumerable.*

- Suppose $\bar{L}$ is recursively enumerable.
- Then both L and $\bar{L}$ are recursively enumerable.
- By Lemma 11, L is recursive, a contradiction.

Corollary 13 *$\bar{H}$ is not recursively enumerable.*

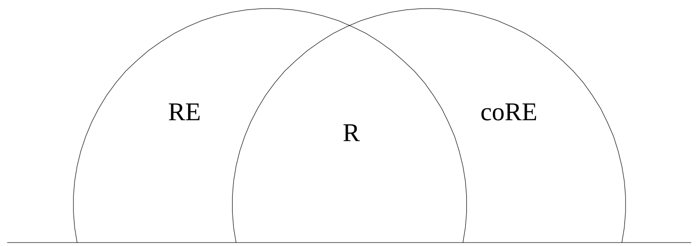
R, RE, and coRE

RE: The set of all recursively enumerable languages.

coRE: The set of all languages whose complements are recursively enumerable (note that coRE is not $\overline{\text{RE}}$).

R: The set of all recursive languages.

- R = RE $\cap$ coRE (p. 120).
- There exist languages in RE but not in R or coRE (such as H).
- There are languages in coRE but not in R or RE (such as $\bar{H}$).
- There are languages in neither RE nor coRE.



Notations

- Suppose M is a TM accepting L.
- Write $L(M) = L$.
- If $M(x)$ is never "yes" nor $\nearrow$ (as required by the definition of acceptance), we define $L(M) = \emptyset$.
- Of course, if $M(x) = \nearrow$ for all x, then $L(M) = \emptyset$, too.

Nontrivial Properties of Sets in RE

- A property of a set accepted by a TM (a recursively enumerable set) is **trivial** if it is always true or false.
  - Is an RE set accepted by a TM? Always true.
- It can be defined by the set $\mathcal{C}$ of RE sets that satisfy it.
- The property is nontrivial if $\mathcal{C} \neq \text{RE}$ and $\mathcal{C} \neq \emptyset$.
- Up to now, all nontrivial properties of RE sets are undecidable (pp. 116–117).
- In fact, Rice's theorem confirms that.

Rice's Theorem

Theorem 14 (Rice's theorem) *Suppose $\mathcal{C} \neq \emptyset$ is a proper subset of the set of all recursively enumerable languages. Then the question "$L(M) \in \mathcal{C}$?" is undecidable.*

- Assume that $\emptyset \not\in \mathcal{C}$ (otherwise, repeat the proof for the class of all recursively enumerable languages *not* in $\mathcal{C}$).
- Let $L \in \mathcal{C}$ be accepted by TM M_L (recall that $\mathcal{C} \neq \emptyset$).
- Let M_H *accept* the undecidable language H.
  - M_H exists (p. 111).

The Proof (continued)

- Construct machine $M_x(y)$:

$$\textbf{if } M_H(x) = \text{"yes" } \textbf{then } M_L(y) \textbf{ else } \nearrow$$

- We next prove that

$$L(M_x) \in \mathcal{C} \text{ if and only if } x \in H. \tag{2}$$

  - The halting problem has been reduced to deciding $L(M_x) \in \mathcal{C}$.
  - Hence $L(M_x) \in \mathcal{C}$ must be undecidable, and we are done.

The Proof (concluded)

- Suppose $x \in H$, i.e., $M_H(x) =$ "yes."
  - $M_x(y)$ determines this, and it either accepts y or never halts, depending on whether $y \in L$.
  - Hence $L(M_x) = L \in \mathcal{C}$.
- Suppose $M_H(x) = \nearrow$.
  - M_x never halts.
  - $L(M_x) = \emptyset \notin \mathcal{C}$.

Consequences of Rice's Theorem

Corollary 15 *The following properties of recursively enumerative sets are undecidable.*

- *Emptiness.*
- *Finiteness.*
- *Regularity.*
- *Context-freedom.*