Comments

- Zero knowledge is a property of the prover.
  - It is the robustness of the prover against attempts of the verifier to extract knowledge via interaction.
  - The verifier may deviate arbitrarily (but in polynomial time) from the predetermined program.
  - A verifier cannot use the transcript of the interaction to convince a third-party of the validity of the claim.
  - The proof is hence not transferable.

Comments (continued)

- Whatever a verifier can "learn" from the specified prover P via the communication channel could as well be computed from the verifier alone.
- The verifier does not learn anything except "$x \in L$."
- For all practical purposes "whatever" can be done after interacting with a zero-knowledge prover can be done by just believing that the claim is indeed valid.
- Zero-knowledge proofs yield no knowledge in the sense that they can be constructed by the verifier who believes the statement, and yet these proofs do convince him.

Comments (concluded)

- The "paradox" is resolved by noting that it is not the transcript of the conversation that convinces the verifier, but the fact that this conversation was held "on line."
- There is no zero-knowledge requirement when $x \notin L$.
- *Computational* zero-knowledge proofs are based on complexity assumptions.
- It is known that if one-way functions exist, then zero-knowledge proofs exist for all problems in NP.

Zero-Knowledge Proof of Quadratic Residuosity

1: **for** $m = 1, 2, \ldots, \log_2 n$ **do**
2: Peggy chooses a random $v \in Z_n^*$ and sends $y = v^2 \bmod n$ to Victor;
3: Victor chooses a random bit i and sends it to Peggy;
4: Peggy sends $z = u^i v \bmod n$, where u is a square root of x; $\{u^2 \equiv x \bmod n.\}$
5: Victor checks if $z^2 \equiv x^i y \bmod n$;
6: **end for**
7: Victor accepts x if Line 5 is confirmed every time;

Analysis

- Assume extracting the square root of a quadratic residue modulo a product of two primes is hard without knowing the factors.
- Suppose x is a quadratic nonresidue.
  - Peggy can answer only one of the two possible challenges.
    * Reason: y is a quadratic residue if and only if xy is a quadratic nonresidue.
  - So Peggy will be caught in any given round with probability one half.

Analysis (continued)

- Suppose x is a quadratic residue.
  - Peggy can answer all challenges.
  - So Victor will accept x.
- How about the claim of zero knowledge?
- The transcript between Peggy and Victor when x is a quadratic residue can be generated without Peggy!
  - So interaction with Peggy is useless.

Analysis (continued)

- Here is how.
- Suppose x is a quadratic residue.
- In each round of interaction with Peggy, the transcript is a triplet (y, i, z).
- We present an efficient algorithm Bob that generates (y, i, z) with the same probability *without* accessing Peggy.

Analysis (concluded)

1: Bob chooses a random $z \in Z_n^*$;

2: Bob chooses a random bit i;

3: Bob calculates $y = z^2 x^{-i} \bmod n$;

4: Bob writes (y, i, z) into the transcript;

Comments

- Bob cheats because (y, i, z) is *not* generated in the same order as in the original transcript.
  - Bob picks Victor's challenge first.
  - Bob then picks Peggy's answer.
  - Bob finally patches the transcript.
  - So it is not the transcript that convinces Victor, but that conversation with Peggy is held "on line."
- The same holds even if the transcript was generated by a cheating Victor's interaction with (honest) Peggy, but we skip the details.

Zero-Knowledge Proof of 3 Colorability[a]

for $i = 1, 2, \ldots, |E|^2$ **do**
  Peggy chooses a random permutation π of the 3-coloring ϕ;
  Peggy samples an encryption scheme randomly and sends $\pi(\phi(1)), \pi(\phi(2)), \ldots, \pi(\phi(|V|))$ encrypted to Victor;
  Victor chooses at random an edge $e \in E$ and sends it to Peggy for the coloring of the endpoints of e;
  if $e = (u, v) \in E$ **then**
    Peggy reveals the coloring of u and v and "proves" that they correspond to their encryption;
  else
    Peggy stops;
  end if

[a] Goldreich, Micali, and Wigderson (1986).

```
10:     if the "proof" provided in Line 6 is not valid then
11:         Victor rejects and stops;
12:     end if
13:     if π(φ(u)) = π(φ(v)) or π(φ(u)), π(φ(v)) ∉ {1, 2, 3} then
14:         Victor rejects and stops;
15:     end if
16: end for
17: Victor accepts;
```

Analysis

- If the graph is 3-colorable and both Peggy and Victor follow the protocol, then Victor always accepts.
- If the graph is not 3-colorable and Victor follows the protocol, then however Peggy plays, Victor will accept with probability $\leq (1 - m^{-1})^{m^2} \leq e^{-m}$, where $m = |E|$.
- Thus the protocol is valid.
- This protocol yields no knowledge to Victor as all he gets is a bunch of random pairs.
- The proof that the protocol is zero-knowledge to *any* verifier is more intricate.

IP and PSPACE

- We next prove that coNP $\subseteq$ IP.
- Shamir in 1990 proved that IP equals PSPACE using similar ideas.

Interactive Proof for Boolean Unsatisfiability

- A 3SAT formula is a conjunction of disjunctions of at most three literals.
- We shall present an interactive proof for boolean unsatisfiability.
- For any unsatisfiable 3SAT formula $\phi(x_1, x_2, \ldots, x_n)$, there is an interactive proof for the fact that it is unsatisfiable.
- Therefore, coNP $\subseteq$ IP.

Arithmetization of Boolean Formulas

The idea is to arithmetize the boolean formula.

- T $\to$ positive integer
- F $\to 0$
- $x_i \to x_i$
- $\bar{x_i} \to 1 - x_i$
- $\vee \to +$
- $\wedge \to \times$
- $\phi(x_1, x_2, \ldots, x_n) \to \Phi(x_1, x_2, \ldots, x_n)$

The Arithmetic Version

- A boolean formula is transformed into a multivariate polynomial Φ.
- It is easy to verify that ϕ is unsatisfiable if and only if

$$\sum_{x_1=0,1} \sum_{x_2=0,1} \cdots \sum_{x_n=0,1} \Phi(x_1, x_2, \ldots, x_n) = 0.$$

- But the above seems to require exponential time.
- We turn to more intricate methods.

Choosing the Field

- Suppose ϕ has m clauses of length three each.
- Then $\Phi(x_1, x_2, \ldots, x_n) \leq 3^m$ for any truth assignment $(x_1, x_2, \ldots, x_n)$.
- Because there are at most 2^n truth assignments,

$$\sum_{x_1=0,1} \sum_{x_2=0,1} \cdots \sum_{x_n=0,1} \Phi(x_1, x_2, \ldots, x_n) \leq 2^n 3^m.$$

Choosing the Field (concluded)

- By choosing a prime $q > 2^n 3^m$ and working modulo this prime, proving unsatisfiability reduces to proving that

$$\sum_{x_1=0,1} \sum_{x_2=0,1} \cdots \sum_{x_n=0,1} \Phi(x_1, x_2, \ldots, x_n) \equiv 0 \bmod q.$$

- Working under a *finite* field allows us to uniformly select a random element in the field.

Binding Peggy

- Peggy has to find a sequence of polynomials that satisfy a number of restrictions.
- The restrictions are imposed by Victor: After receiving a polynomial from Peggy, Victor sets a new restriction for the next polynomial in the sequence.
- These restrictions guarantee that if ϕ is unsatisfiable, such a sequence can always be found.
- However, if ϕ is not unsatisfiable, any Peggy has only a small probability of finding such a sequence.
  - The probability is taken over Victor's coin tosses.

The Algorithm

Peggy and Victor both arithmetize ϕ to obtain Φ;
Peggy picks a prime $q > 2^n 3^m$ and sends it to Victor;
Victor rejects and stops if q is not a prime;
Victor sets v_0 to 0;
for $i = 1, 2, \ldots, n$ **do**
    Peggy calculates $P_i^*(z) = \sum_{x_{i+1}=0,1} \cdots \sum_{x_n=0,1} \Phi(r_1, \ldots, r_{i-1}, z, x_{i+1}, \ldots, x_n)$;
    Peggy sends $P_i^*(z)$ to Victor;
    Victor rejects and stops if $P_i^*(0) + P_i^*(1) \not\equiv v_{i-1} \bmod q$ or $P_i^*(z)$'s degree exceeds m; {$P_i^*(z)$ has at most m clauses.}
    Victor uniformly picks $r_i \in Z_q$ and sets $v_i = P_i^*(r_i) \bmod q$;
    Victor sends r_i to Peggy;
end for
Victor accepts iff $\Phi(r_1, r_2, \ldots, r_n) \equiv v_n \bmod q$;

Comments

- The following invariant is maintained by the algorithm:

$$P_i^*(0) + P_i^*(1) \equiv P_{i-1}^*(r_{i-1}) \bmod q \tag{11}$$

  for $1 \leq i \leq n$.

- The computation of $v_1, v_2, \ldots, v_n$ must rely on Peggy's supplied polynomials as Victor does not have the power to carry out the exponential-time calculations.

- But $\Phi(r_1, r_2, \ldots, r_n)$ in Step 12 is computed without relying on Peggy's polynomials.

Completeness

- Suppose ϕ is unsatisfiable.
- For $i \geq 1$,

$$\begin{aligned} & P_i^*(0) + P_i^*(1) \\ = & \sum_{x_i=0,1} \cdots \sum_{x_n=0,1} \Phi(r_1, \ldots, r_{i-1}, x_i, \ldots, x_n) \\ = & P_{i-1}^*(r_{i-1}) \\ \equiv & v_{i-1} \bmod q. \end{aligned}$$

Completeness (concluded)

- In particular at $i = 1$, because ϕ is unsatisfiable, we have

$$\begin{aligned} P_1^*(0) + P_1^*(1) &= \sum_{x_1=0,1} \cdots \sum_{x_n=0,1} \Phi(x_1, \ldots, x_n) \\ &\equiv v_0 \\ &= 0 \bmod q. \end{aligned}$$

- Finally, $v_n = P_n^*(r_n) = \Phi(r_1, r_2, \ldots, r_n)$.
- Because all the tests by Victor will pass, Victor will accept ϕ.

Soundness

- Suppose ϕ is not unsatisfiable.
- An honest Peggy following the protocol will fail after sending $P_1^*(z)$.
- We will show that if Peggy is dishonest in one round (by sending a polynomial other than $P_i^*(z)$), then with high probability she must be dishonest in the next round, too.
- In the last round (Step 12), her dishonesty is exposed.

Soundness (continued)

- Let $P_i(z)$ represent the polynomial sent by Peggy in place of $P_i^*(z)$.
- Victor calculates $v_i = P_i(r_i) \bmod p$.
- In order to deceive Victor in the next round, round $i+1$, Peggy must use $r_1, r_2, \ldots, r_i$ to find a $P_{i+1}(z)$ of degree at most m such that

$$P_{i+1}(0) + P_{i+1}(1) = v_i \bmod q$$

(see Step 8 of the algorithm on p. 526).

- And so on to the end, except that Peggy has no control over Step 12.

A Key Claim

Theorem 82 *If $P_i^*(0) + P_i^*(1) \not\equiv v_{i-1} \bmod q$, then either Victor rejects in the ith round, or $P_i^*(r_i) \not\equiv v_i \bmod q$ with probability at least $1 - (m/q)$, where the probability is taken over Victor's choices of r_i.*

- Remember that Victor has no way of knowing $P_i^*(r_i)$.
- Victor calculates v_i's with $P_i(z)$s, claimed by the not necessarily trust-worthy Peggy as $P_i^*(z)$s.
- What Victor can do is to check for consistencies.

The Proof of Theorem 82 (continued)

- If Peggy sends a $P_i(z)$ which equals $P_i^*(z)$, then

$$P_i(0) + P_i(1) = P_i^*(0) + P_i^*(1) \not\equiv v_{i-1} \bmod q,$$

  and Victor rejects immediately.

- Suppose Peggy sends a $P_i(z)$ different from $P_i^*(z)$.
- If $P_i(z)$ does not pass Victor's test

$$P_i(0) + P_i(1) \equiv v_{i-1} \bmod q, \tag{12}$$

  then Victor rejects and we are done, too.

The Proof of Theorem 82 (concluded)

- Finally, assume $P_i(z)$ passes the test (12).
- Because $P_i(z) - P_i^*(z) \not\equiv 0$ is a polynomial of degree at most m, it has at most m roots $r_i \in Z_q$, i.e.,

$$P_i^*(r_i) \equiv v_i \bmod q.$$

- Hence

$$P_i^*(r_i) \equiv v_i \bmod q$$

with probability at most m/q.

Soundness (continued)

- Suppose Victor does not reject in any of the first n rounds.
- As ϕ is not unsatisfiable,

$$P_1^*(0) + P_1^*(1) \not\equiv v_0 \bmod q.$$

- By Theorem 82 (p. 532) and the fact that Victor does not reject, we have $P_1^*(r_1) \not\equiv v_1 \bmod q$ with probability at least $1 - (m/q)$.
- Now by Eq. (11) on p. 527,

$$P_1^*(r_1) = P_2^*(0) + P_2^*(1) \not\equiv v_1 \bmod q.$$

Soundness (concluded)

- Iterating on this procedure, we eventually arrive at

$$P_n^*(r_n) \not\equiv v_n \bmod q$$

with probability at least $(1 - m/q)^n$.

- As $P_n^*(r_n) = \Phi(r_1, r_2, \ldots, r_n)$, Victor's last test at Step 12 fails and he rejects.

- Altogether, Victor rejects with probability at least

$$[\,1 - (m/q)\,]^n > 1 - (nm/q) > 2/3$$

because $q > 2^n 3^m$.

An Example

- $(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3)$.
- The above is satisfied by assigning true to x_1.
- The arithmetized formula is

  $$\Phi(x_1, x_2, x_3) = (x_1 + x_2 + x_3) \times [\, x_1 + (1 - x_2) + (1 - x_3) \,].$$

- Indeed, $\sum_{x_1=0,1} \sum_{x_2=0,1} \sum_{x_3=0,1} \Phi(x_1, x_2, x_3) = 16 \neq 0$.
- We have $n = 3$ and $m = 2$.
- A prime q that satisfies $q > 2^3 \times 3^2 = 72$ is 73.

An Example (continued)

- The table below is an execution of the algorithm in Z_{73} *when Peggy follows the protocol.*

i	$P_i^*(z)$	$P_i^*(0) + P_i^*(1)$	$= v_{i-1}$?	r_i	v_i
0					0
1	$4z^2 + 8z + 2$	16	no		

- Victor therefore rejects ϕ early on at $i = 1$.

An Example (continued)

- Now suppose Peggy does not follow the protocol.
- In order to deceive Victor, she comes up with fake polynomials $P_i(z)$'s from beginning to end.
- The table below is an execution of the algorithm.

i	$P_i(z)$	$P_i(0) + P_i(1)$	$= v_{i-1}$?	r_i	v_i
0					0
1	$8z^2 + 11z + 27$	0	yes	10	61
2	$10z^2 + 9z + 21$	61	yes	4	71
3	$z^2 + 2z + 34$	71	yes	r_3	$P_3(r_3)$

An Example (concluded)

- Victor has been satisfied up to round 3.
- Finally at Step 12, Victor checks if

$$\Phi(10, 4, r_3) \equiv P_3(r_3) \bmod 73.$$

- It can be verified that the only choices of $r_3 \in \{0, 1, \ldots, 72\}$ that can mislead Victor are 10 and 12.
- The probability of that happening is only 2/73.

An Example

- $(x_1 \vee x_2) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2)$.
- The above is unsatisfiable.
- The arithmetized formula is

  $$\Phi(x_1, x_2) = (x_1 + x_2) \times (x_1 + 1 - x_2) \times (1 - x_1 + x_2) \times (2 - x_1 - x_2).$$

- Because $\Phi(x_1, x_2) = 0$ for any *boolean* assignment $\{0, 1\}^2$ to (x_1, x_2), certainly

  $$\sum_{x_1=0,1} \sum_{x_2=0,1} \Phi(x_1, x_2) = 0.$$

- With $n = 2$ and $m = 4$, a prime q that satisfies $q > 2^2 \times 3^4 = 4 \times 81 = 324$ is 331.

An Example (concluded)

- The table below is an execution of the algorithm in Z_{331}.

i	$P_i^*(z)$	$P_i^*(0) + P_i^*(1)$	$= v_{i-1}$?	r_i	v_i
0					0
1	$z(z+1)(1-z)(2-z)$ $+(z+1)z(2-z)(1-z)$	0	yes	10	283
2	$(10+z) \times (11-z)$ $\times(-9+z) \times (-8-z)$	283	yes	5	46

- Victor calculates $\Phi(10,5) \equiv 46 \bmod 331$.

- As it equals $v_2 = 46$, Victor accepts ϕ as unsatisfiable.

Objections to the Soundness Proof?[a]

- Based on the steps required of a cheating Peggy on p. 531, why must we go through so many rounds (in fact, n rounds)?
- Why not just go directly to round n:
  - Victor sends $r_1, r_2, \ldots, r_{n-1}$ to Peggy.
  - Peggy returns with a (claimed) $P_n^*(z)$.
  - Victor accepts if and only if $\Phi(r_1, r_2, \ldots, r_{n-1}, r_n) \equiv P_n^*(r_n) \bmod q$ for a random $r_n \in Z_q$.

[a]Contributed by Ms. Emily Hou (D89011) and Mr. Pai-Hsuen Chen (R90008) on January 2, 2002.

Objections to the Soundness Proof? (continued)

- Let us analyze the compressed proposal when ϕ is satisfiable.
- To succeed in foiling Victor, Peggy must find a polynomial $P_n(z)$ of degree m such that

$$\Phi(r_1, r_2, \ldots, r_{n-1}, z) \equiv P_n(z) \bmod q.$$

- But this she is able to do: Just give the verifier the polynomial $\Phi(r_1, r_2, \ldots, r_{n-1}, z)$!
- What has happened?

Objections to the Soundness Proof? (concluded)

- You need the intermediate rounds to "tie" Peggy up with a chain of claims.
- In the original algorithm on p. 526, for example, $P_n(z)$ is bound by the equality $P_n(0) + P_n(1) \equiv v_{n-1} \bmod q$ in Step 8.
- That v_{n-1} is in turn derived by an earlier polynomial $P_{n-1}(z)$, which is in turn bound by $P_{n-1}(0) + P_{n-1}(1) \equiv v_{n-2} \bmod q$, and so on.

Density[a]

The **density** of language $L \subseteq \Sigma^*$ is defined as

$$\text{dens}_L(n) = |\{x \in L : |\,x\,| \leq n\}|.$$

- If $L = \{0,1\}^*$, then $\text{dens}_L(n) = 2^{n+1} - 1$.
- So the density function grows at most exponentially.
- For a unary language $L \subseteq \{0\}^*$,

$$\text{dens}_L(n) \leq n + 1.$$

  – Because $L \subseteq \{0, 00, \ldots, \overbrace{00 \cdots 0}^{n}, \ldots\}$.

[a]Berman and Hartmanis (1977).

Sparsity

- **Sparse languages** are languages with polynomially bounded density functions.
- **Dense languages** are languages with superpolynomial density functions.

Self-Reducibility for SAT

- An algorithm exploits **self-reducibility** if it reduces the problem to the same problem with a smaller size.
- Let ϕ be a boolean expression in n variables $x_1, x_2, \ldots, x_n$.
- $t \in \{0,1\}^j$ is a **partial** truth assignment for $x_1, x_2, \ldots, x_j$.
- $\phi[\,t\,]$ denotes the expression after substituting the truth values of t for $x_1, x_2, \ldots, x_{|\,t\,|}$ in ϕ.

An Algorithm for SAT with Self-Reduction

We call the algorithm below with empty t.

if $|\, t \,| = n$ **then**

  return $\phi[\, t \,]$;

else

  return $\phi[\, t0 \,] \lor \phi[\, t1 \,]$;

end if

The above algorithm runs in exponential time.

NP-Completeness and Density[a]

Theorem 83 *If a unary language* $U \subseteq \{0\}^*$ *is NP-complete, then* $P = NP$.

- Suppose there is a reduction R from SAT to U.
- We shall use R to guide us in finding the truth assignment that satisfies a given boolean expression ϕ with n variables if it is satisfiable.
- Specifically, we use R to prune the exponential-time exhaustive search on p. 549.
- The trick is to keep the already discovered results $\phi[\,t\,]$ in a hash table H.

---

[a]Berman (1978).

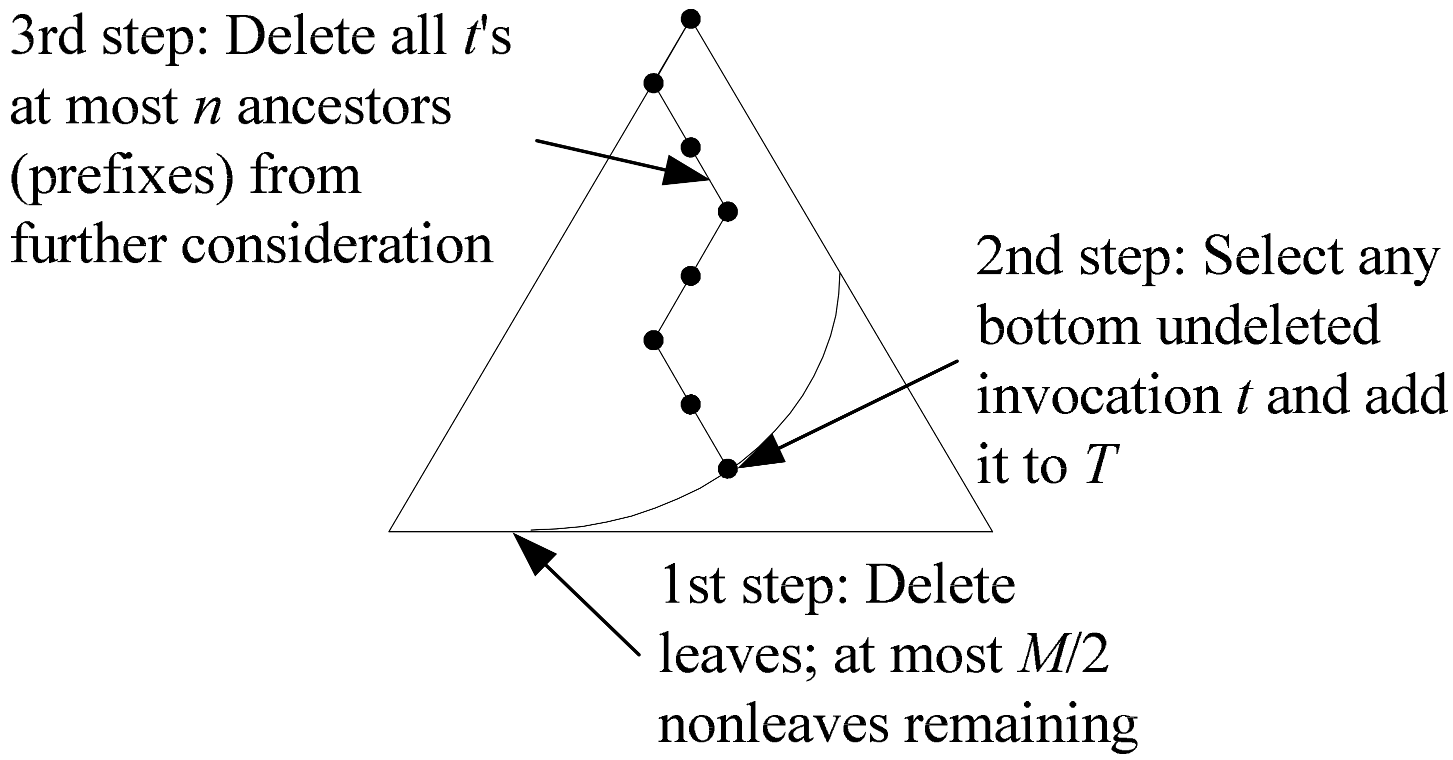
```
if |t| = n then
    return φ[t];
else
    if (R(φ[t]), v) is in table H then
        return v;
    else
        if φ[t0] = "satisfiable" or φ[t1] = "satisfiable" then
            Insert (R(φ[t]), 1) into H;
            return "satisfiable";
        else
            Insert (R(φ[t]), 0) into H;
            return "unsatisfiable";
        end if
    end if
end if
```

The Proof (continued)

- Since R is a reduction, $R(\phi[\,t\,]) = R(\phi[\,t'\,])$ implies that $\phi[\,t\,]$ and $\phi[\,t'\,]$ must be both satisfiable or unsatisfiable.
- $R(\phi[\,t\,])$ has polynomial length $\leq p(n)$ because R runs in log space.
- As R maps to unary numbers, there are only polynomially many $p(n)$ values of $R(\phi[\,t\,])$.
- How many nodes of the complete binary tree (of invocations/truth assignments) need to be visited?
- If that number is a polynomial, the overall algorithm runs in polynomial time and we are done.

The Proof (continued)

- A search of the table takes time $O(p(n))$ in the random access memory model.
- The running time is $O(Mp(n))$, where M is the total number of invocations of the algorithm.
- The invocations of the algorithm form a binary tree of depth at most n.
- There is a set $T = \{t_1, t_2, \ldots\}$ of invocations (partial truth assignments, i.e.) such that:
  - $|T| \geq M/(2n)$.
  - All invocations in T are **recursive** (nonleaves).
  - None of the elements of T is a prefix of another.


3rd step: Delete all t's at most n ancestors (prefixes) from further consideration
2nd step: Select any bottom undeleted invocation t and add it to T
1st step: Delete leaves; at most M/2 nonleaves remaining

The Proof (continued)

- All invocations $t \in T$ have different $R(\phi[\,t\,])$ values.
  - None of $s, t \in T$ is a prefix of another.
  - The invocation of one started after the invocation of the other had terminated.
  - If they had the same value, the one that was invoked second would have looked it up, and therefore would not be recursive, a contradiction.
- The existence of T implies that there are at least $M/(2n)$ different $R(\phi[\,t\,])$ values in the table.

The Proof (concluded)

- We already know that there are at most $p(n)$ such values.
- Hence $M/(2n) \leq p(n)$.
- Thus $M \leq 2np(n)$.
- The running time is therefore $O(Mp(n)) = O(np^2(n))$.
- We comment that this theorem holds for any sparse language, not just unary ones.[a]

[a] Mahaney (1980).

NP-Completeness and Density

Theorem 84 (Fortung (1979)) *If a unary language $U \subseteq \{0\}^*$ is coNP-complete, then P = NP.*

- Suppose there is a reduction R from SAT COMPLEMENT to U.
- The rest of the proof is basically identical except that, now, we want to make sure a formula is unsatisfiable.

Finis