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LINEAR-TIME SUCCINCT ENCODINGS OF PLANAR GRAPHS VIA CANONICAL ORDERINGS*

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Abstract. Let G be an embedded planar undirected graph that has n vertices, m edges, and f faces but has no self-loop or multiple edge. If G is triangulated, we can encode it using $\frac{4}{3}m - 1$ bits, improving on the best previous bound of about $1.53m$ bits. In case exponential time is acceptable, roughly $1.08m$ bits have been known to suffice. If G is triconnected, we use at most $(2.5 + 2\log 3)\min\{n, f\} - 7$ bits, which is at most $2.835m$ bits and smaller than the best previous bound of $3m$ bits. Both of our schemes take $O(n)$ time for encoding and decoding.

Key words. data compression, graph encoding, canonical ordering, planar graphs, triconnected graphs, triangulations

AMS subject classifications. 05C30, 05C78, 05C85, 68R10

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1. Introduction.

This paper investigates the problem of *encoding* a given graph G into a binary string S with the requirement that S can be *decoded* to reconstruct G. The problem has been studied generally with two primary objectives. One is to minimize the length of S, while the other is to minimize the time needed to compute and decode S. In light of these goals, a coding scheme is *efficient* if its encoding and decoding procedures both take polynomial time. A coding scheme is *succinct* if the length of S is not much larger than its *information-theoretic tight bound*, i.e., the shortest length over all possible coding schemes.

As the two primary objectives are often in conflict, a number of coding schemes with different trade-offs have been proposed from practical and theoretical perspectives. The most well known efficient succinct scheme is the folklore scheme of encoding a rooted-ordered n-vertex tree into a string of balanced $n-1$ pairs of left and right parentheses, which uses $2(n-1)$ bits. Since the total number of such trees is at least $\frac{1}{2(n-1)} \cdot \frac{(2n-2)!}{(n-1)!(n-1)!}$, the minimum number of bits needed to differentiate these trees is the logarithm[1] of this quantity, which is $2n - o(n)$ by Stirling's approximation. Thus, two bits per edge is an information-theoretic tight bound for encoding rooted-ordered trees. The standard adjacency-list encoding of a graph is widely useful but requires $\Theta(m\log n)$ bits, where m and n are the numbers of edges and vertices, respectively [3]. For certain graph families, Kannan, Naor, and Rudich [10] gave schemes that encode each vertex with $O(\log n)$ bits and support $O(\log n)$-time testing of adjacency between two vertices. For connected planar graphs, Jacobson [9] gave an $\Theta(n)$-bit encoding which supports traversal in $\Theta(\log n)$ time per vertex visited. This result was recently improved by Munro and Raman [17]; their schemes encode binary trees,

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[1] All logarithms are of base 2.

rooted-ordered trees, and planar graphs succinctly and support several graph operations in constant time. For dense graphs and complement graphs, Kao, Occhiogrosso, and Teng [14] devised two compressed representations from adjacency lists to speed up basic graph techniques such as breadth-first search and depth-first search. Galperin and Wigderson [6] and Papadimitriou and Yannakakis [19] investigated complexity issues arising from encoding a graph by a small circuit that computes its adjacency matrix. For labeled planar graphs, Itai and Rodeh [8] gave an encoding procedure that requires $\frac{3}{2}n \log n + O(n)$ bits. For unlabeled general graphs, Naor [18] gave an encoding of $\frac{n^2}{2} - n \log n + O(n)$ bits, which is optimal to the second order.

Our work aims to minimize the number of bits needed to encode an embedded planar graph G which is unlabeled and undirected. We assume that G has n vertices, m edges, and f faces but has no self-loop nor multiple edge. (See [2, 3, 7, 16] for the graph-theoretic terminology used in this paper.) Note that if polynomial time for encoding and decoding is not required, then any given graph in a large family can be encoded with the information-theoretic minimum number of bits by brute-force enumeration. This paper focuses on schemes that use only $O(n)$ time for both encoding and decoding.

For a general planar graph G, Turán [21] gave an encoding using $4m$ bits asymptotically. This space complexity was improved by Keeler and Westbrook [15] to about $3.58m$ bits. They also gave encoding algorithms for several important classes of planar graphs. In particular, they showed that if G is triangulated, it can be encoded in about $1.53m$ bits. If G is triconnected, it can be encoded using $3m$ bits. In this paper, these latter two results are improved as follows. If G is triangulated, it can be encoded using $\frac{4}{3}m - 1$ bits. It is interesting that rooted-ordered trees require two bits per edge, while the seemingly more complex plane triangulations need fewer bits. Note that Tutte [22] gave an enumeration theorem that yields an information-theoretic tight bound of roughly $1.08m$ bits for plane triangulations that may contain multiple edges. If G is triconnected, we can encode it using at most $(2.5 + 2 \log 3) \min\{n, f\} - 7$ bits, which is at most $2.835m$ bits. Both of our coding schemes are intuitive and simple. They require only $O(n)$ time for encoding as well as decoding. The schemes make new uses of the canonical orderings of planar graphs, which were originally introduced by de Fraysseix, Pach, and Pollack [4] and extended by Kant [11]. These structures and closely related ones have proven useful also for drawing planar graphs in organized and compact manners [12, 13, 20].

This paper is organized as follows. In section 2, we present our coding scheme for plane triangulations. In section 3, we generalize the scheme to encode triconnected plane graphs. We conclude the paper with some open problems in section 4.

2. A coding scheme for plane triangulations.

This section assumes that G is a plane triangulation. Thus, $n \geq 3$ and G has $m = 3n - 6$ edges.

Let $v_1, \ldots, v_n$ be an ordering of the vertices of G, where v_1, v_2, v_n are the three exterior vertices of G in the counterclockwise order. After fixing such an ordering, let G_k be the subgraph of G induced by $v_1, \ldots, v_k$. Let H_k be the exterior face of G_k. Let $G - G_k$ be the subgraph of G obtained by removing $v_1, \ldots, v_k$. Our coding scheme uses a special kind of ordering defined as follows.

DEFINITION 2.1 (see [4]). *An ordering $v_1, \ldots, v_n$ of G is* canonical *if the following statements hold for every $k = 3, \ldots, n$:*

1. *G_k is biconnected, and its exterior face H_k is a cycle containing the edge (v_1, v_2).*
2. *The vertex v_k is on the exterior face of G_k, and the set of its neighbors in*

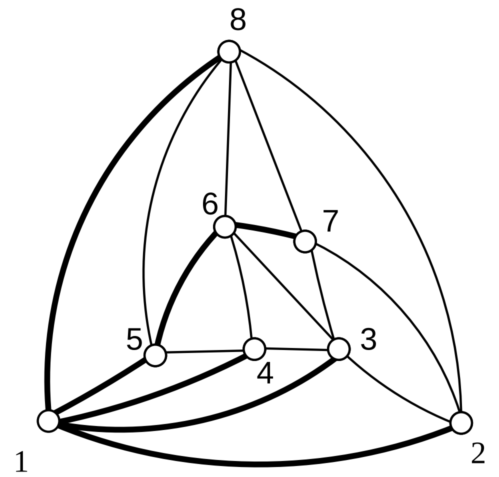


FIG. 2.1. *A plane triangulation and a canonical ordering.*

G_{k-1} forms a subinterval of the path $H_{k-1} - \{(v_1, v_2)\}$ and consists of at least two vertices. Furthermore, if $k < n$, v_k has at least one neighbor in $G - G_k$. Note that the case $k = 3$ is somewhat ambiguous due to degeneracy, and $H_2 - \{(v_1, v_2)\}$ is regarded as the edge (v_1, v_2) itself.

Figure 2.1 illustrates a canonical ordering of a plane triangulation. Note that every plane triangulation has a canonical ordering which can be computed in $O(n)$ time [4]. A canonical ordering of G can be viewed as an order in which G is reconstructed from a single edge (v_1, v_2) step by step. At step k with $3 \leq k \leq n$, the vertex v_k and the edges between v_k and its lower ordered neighbors are added into the graph. For the sake of enhancing intuitions, we call H_{k-1} the *contour* of G_{k-1}; denote its vertices by $c_1(= v_1), c_2, \ldots, c_{t-1}, c_t(= v_2)$ in the consecutive order along the cycle H_{k-1}; and visualize them as arranged from left to right above the edge (v_1, v_2) in the plane. When the vertex v_k is added to G_{k-1} to construct G_k, let $c_\ell, c_{\ell+1}, \ldots, c_r$ be the neighbors of v_k on the contour H_{k-1}. After v_k is added, the vertices $c_{\ell+1}, \ldots, c_{r-1}$ are no longer contour vertices. Thus, we say that these vertices are *covered* by v_k. The edge (v_k, c_ℓ) is the *left* edge of v_k, the edge (v_k, c_r) is the *right* edge of v_k, and the edges (c_p, v_k) with $\ell < p < r$ are the *internal* edges of v_k.

There is no published reference for the following folklore lemma; for the sake of completeness, we include its proof here.

LEMMA 2.2. *Let $v_1, \ldots, v_n$ be a canonical ordering of G. Let T_1 (respectively, T_2) be the collection of the left (respectively, right) edges of v_j for $3 \leq j \leq n-1$; similarly, let T_n be that of the internal edges of v_j for $3 \leq j \leq n$.*

1. *T_1 is a tree spanning over $G - \{v_2, v_n\}$.*
2. *T_2 is a tree spanning over $G - \{v_1, v_n\}$.*
3. *T_n is a tree spanning over $G - \{v_1, v_2\}$.*

Proof. The statements are proved separately as follows.

Statement 1. For $i = 3, \ldots, n-1$, let D_i be the collection of the left edges of v_j for $3 \leq j \leq i$. We prove by induction on i the claim that D_i is a tree spanning over $v_1, v_3, \ldots, v_i$. Then, since $T_1 = D_{n-1}$, the claim implies the statement. For the base case $i = 3$, the claim trivially holds. The induction hypothesis is that the claim holds for $i = k-1 < n-1$. The induction step is to prove the claim for $i = k \leq n-1$.

D_k is obtained from D_{k-1} by adding the left edge (v_k, c_ℓ) of v_k. By the induction hypothesis, D_{k-1} is a tree spanning over $v_1, v_3, \ldots, v_{k-1}$. Since c_ℓ is the left-most neighbor of v_k on H_{k-1}, c_ℓ is some v_j with $1 \leq j \leq k-1$ and $j \neq 2$. Thus, D_{k-1} contains c_ℓ, and D_k is a tree spanning over $v_1, v_3, \ldots, v_{k-1}, v_k$.

Statement 2. The proof is symmetric to that of Statement 1.

Statement 3. G has n vertices and $3n-6$ edges. The edges $(v_1, v_2), (v_2, v_n), (v_1, v_n)$ are not in $T_1 \cup T_2 \cup T_n$. Thus, since T_1 and T_2 have $n-3$ edges each, T_n has $n-3$ edges. Then, since T_n is acyclic and does not contain v_1 and v_2, T_n is a spanning tree of $G - \{v_1, v_2\}$. ◻

A canonical ordering $v_1, \ldots, v_n$ is *right-most* if for all v_k and $v_{k'}$ with $k' > k$ such that the neighbors of $v_{k'}$ on $H_{k'-1}$ are all in H_{k-1}, the *left-most* neighbor of $v_{k'}$ appears before that of v_k when traversing H_{k-1} from v_1 to v_2 clockwise. Intuitively speaking, if there are more than one vertex that can be added to G_{k-1}, we always add the right-most one. The ordering in Figure 2.1 is right-most. A right-most canonical ordering is symmetric to a left-most one in [11] and can be computed from G in linear time similarly.

Let $v_1, \ldots, v_n$ be a right-most canonical ordering of G. Let T_1 be as in Lemma 2.2 for this ordering. Let T be the tree $T_1 \cup \{(v_1, v_n), (v_1, v_2)\}$. In Figure 2.1, T is indicated by the thick lines. Our coding scheme uses T extensively. The *right-most* depth-first search of T proceeds as follows. We start at v_1 and traverse the edge (v_1, v_2) first. Afterward, if two or more vertices can be visited from v_k, we choose the right-most one. More precisely, let P be the path in T from v_k to v_1 and then to v_2. Let D be the set of edges between v_k and the available vertices. We visit a new vertex through the edge in D that is next to P in the counterclockwise cyclic order around v_k formed by P and the edges in D. Note that the order in which the vertices are visited by the right-most depth-first search is the right-most canonical ordering $v_1, \ldots, v_n$ that defines T.

We are now ready to describe the encoding S of G as the concatenation of two binary strings S_1 and S_2 as follows.

S_1 is the binary string that encodes T using the folklore parenthesis coding scheme where 0 and 1 correspond to "(" and ")", respectively. In this encoding, T is rooted at v_1, and the branches are ordered the same as their endpoints are in the right-most canonical ordering. Since T contains n vertices, S_1 has $2(n-1)$ bits.

S_2 encodes the number of contour vertices covered by each v_k with $3 \leq k \leq n$. First, we create a string of $n-2$ copies of 0. The $(k-2)$th 0 corresponds to v_k. If v_k covers d vertices, we insert d copies of 1 before the corresponding 0. For example, the string S_2 for Figure 2.1 is

$$00010101110.$$

Since each vertex v_k with $3 \leq k \leq n-1$ is covered exactly once, S_2 has $n-3$ copies of 1. So $|S_2| = (n-2) + (n-3) = 2n-5$ bits. Hence, $|S| = |S_1| + |S_2| = 4n-7$ bits.

We next describe how to decode S to reconstruct G. Given S, we can uniquely determine n from the length of S. Subsequently, we can uniquely determine S_1 and S_2. From S_1, we can reconstruct T. From T, we can recover the ordering $v_1, \ldots, v_n$. Then, we draw the edge (v_1, v_2) and perform a loop of $n-2$ steps indexed by k with $3 \leq k \leq n$, where step k processes v_k. Before v_k is processed, G_{k-1} and its contour H_{k-1} have been constructed. At step k, we add v_k and the edges between v_k and its lower ordered neighbors into G_{k-1} to construct G_k as follows. From T, we can identify the left-most neighbor c_ℓ of v_k on the contour H_{k-1} because c_ℓ is simply the parent of v_k in T. From S_2, we can determine the number d of vertices covered by v_k. Thus, we add the edges $(c_\ell, v_k), (c_{\ell+1}, v_k), \ldots, (c_{\ell+d+1}, v_k)$ into G_{k-1}; note that $r = \ell + d + 1$. This gives us the subgraph G_k and completes step k.

It is straightforward to carry out these encoding and decoding procedures in linear time. Also, we can save one bit by deleting the last 0 in S_2. Since v_3 covers no vertex, for $n \geq 4$, we can save another bit by deleting the first 0 in S_2. Note that for $n = 3$, the last 0 in S_2 is also the first 0 and cannot be deleted twice, but we can simply encode the 3-vertex plane triangulation with zero bit without ambiguity. Thus, we have the following theorem.

THEOREM 2.3. *A plane triangulation of m edges and n vertices with $n \geq 4$ can be encoded using $4n - 9 = \frac{4}{3}m - 1$ bits. Both encoding and decoding take $O(n)$ time.*

3. A coding scheme for triconnected plane graphs. This section assumes that G is triconnected. To avoid triviality, let $n \geq 3$.

Let $v_1, \ldots, v_n$ be an ordering of the vertices of G, where v_1, v_2, v_n are on the exterior face of G, and v_2 and v_n are neighbors of v_1. Let G_k be the subgraph of G induced by $v_1, \ldots, v_k$. Let H_k be the exterior face of G_k. Let $G - G_k$ be the subgraph of G obtained by removing $v_1, \ldots, v_k$. Our coding scheme for triconnected plane graphs uses an ordering defined as follows.

DEFINITION 3.1 (see [11]). *An ordering $v_1, \ldots, v_n$ of a triconnected plane graph G is* canonical *if the integer interval $[3, n]$ can be partitioned into subintervals $[k, k+q]$ each satisfying either set of properties below:*

1. *The integer q is 0. The vertex v_k is on the exterior face of G_k and has at least two neighbors in G_{k-1}. G_k is biconnected and its exterior face contains the edge (v_1, v_2). If $k < n$, v_k has at least one neighbor in $G - G_k$.*
2. *The integer q is at least 1. The sequence $v_k, v_{k+1}, \ldots, v_{k+q}$ is a chain on the exterior face of G_{k+q} and has exactly two neighbors in G_{k-1}, one for v_k and the other for v_{k+q}, which are on the exterior face of G_{k-1}. G_{k+q} is biconnected and its exterior face contains the edge (v_1, v_2). Every vertex among $v_k, \ldots, v_{k+q}$ has at least one neighbor in $G - G_{k+q}$.*

As in section 2, we similarly define a right-most canonical ordering $v_1, \ldots, v_n$ of G. Figure 3.1 shows a right-most canonical ordering of a triconnected plane graph. Given a triconnected plane graph, we can find a right-most canonical ordering in linear time [11]. With a right-most canonical ordering, G can be reconstructed from a single edge (v_1, v_2) through a sequence of steps indexed by k'. There are two possible cases at step k', which correspond to the two sets of properties in Definition 3.1 and are used throughout this section.

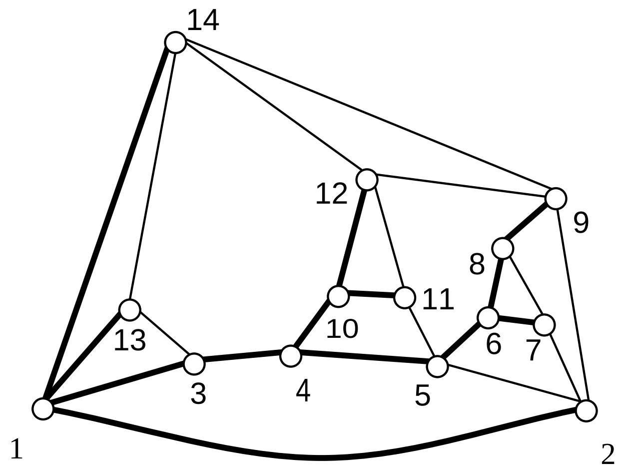
Case 1. A single vertex v_k is added.

Case 2. A chain of $q + 1$ vertices $v_k, \ldots, v_{k+q}$ is added.

While reconstructing G, we collect a set T of edges as follows. Initially, T consists of the edge (v_1, v_2). Let $c_1 (= v_1), c_2, \ldots, c_{t-1}, c_t (= v_2)$ be the vertices of H_{k-1}, which are ordered consecutively along the boundary cycle of H_{k-1} and are arranged from left to right above the edge (v_1, v_2) in the plane.

Case 1. Let c_ℓ and c_r with $1 \leq \ell < r \leq t$ be the left-most and right-most neighbors of v_k in H_{k-1}, respectively. After v_k is added, $c_{\ell+1}, \ldots, c_{r-1}$ are no longer contour vertices; these vertices are *covered* at step k'. The edge (c_ℓ, v_k) is included in T.

Case 2. Let c_ℓ and c_r with $1 \leq \ell < r \leq t$ be the neighbors of v_k and v_{k+q} in H_{k-1}, respectively. After $v_k, \ldots, v_{k+q}$ are added, $c_{\ell+1}, \ldots, c_{r-1}$ are no longer contour vertices; these vertices are covered at step k'. The edges $(c_\ell, v_k), (v_k, v_{k+1}), \ldots, (v_{k+q-1},$



Step:	Vertices added:
1	3, 4, 5
2	6, 7
3	8
4	9
5	10, 11
6	12
7	13
8	14

FIG. 3.1. *A triconnected plane graph and a canonical ordering.*

v_{k+q}) are included in T.

In Figure 3.1, the edges in T are indicated by the thick lines. By an argument similar to the proof of Lemma 2.2, Statement 1, T is a spanning tree of G. As in section 2, we similarly define the *right-most* depth-first search in T. Note that the order in which the vertices of T are visited by the right-most depth-first search is the right-most canonical ordering $v_1, \ldots, v_n$ that defines T.

We are now ready to describe the encoding S of G by means of T. We further divide Case 1 into three subcases.

Case 1a. No vertex is covered at step k'.

Case 1b. At least one vertex is covered at step k' and the left-most covered vertex $c_{\ell+1}$ is adjacent to v_k.

Case 1c. At least one vertex is covered at step k' and the left-most covered vertex $c_{\ell+1}$ is not adjacent to v_k.

Let β be the number of steps for reconstructing G. Let $\beta_{1a}, \beta_{1b}, \beta_{1c}$, and β_2 be the numbers of steps of Cases 1a, 1b, 1c, and 2, respectively. We first consider the case $\beta_{1b} \geq \beta_{1c}$ to encode G with *Scheme* I; afterwards, we modify Scheme I into *Scheme* II for the case $\beta_{1b} < \beta_{1c}$.

In Scheme I, the encoding S of G is the concatenation of three strings S_1, S_2, and S_3. S_1 is the folklore parentheses encoding of T, which is rooted and ordered in the same way as in section 2. Since T has n vertices, S_1 has $2(n-1)$ bits.

To construct S_2, first let $Q = s_1 * s_2 * \cdots * s_\beta *$, where each $s_{k'}$ is a binary string that corresponds to the step k' of reconstructing G based on the ordering $v_1, \ldots, v_n$. $s_{k'}$ is determined as follows. The following two cases both assume that d vertices are covered at step k'.

Case 1. Note that $d = r - \ell - 1$. The string $s_{k'}$ has d symbols corresponding to c_j with $j = \ell+1, \ldots, r-1$, respectively. If the edge (c_j, v_k) is present in G, the symbol in $s_{k'}$ corresponding to c_j is 1; otherwise, the symbol is 0. Note that in Case 1a, since no vertex is covered, $s_{k'}$ is empty.

Case 2. The string $s_{k'}$ consists of q copies of 0 followed by d copies of 1. For example, the string Q for Figure 3.1 is

$$\underbrace{00}_{\substack{\uparrow\\ s_1}} \; * \; \underbrace{0}_{\substack{\uparrow\\ s_2}} \; * \; * \; \underbrace{0}_{\substack{\uparrow\\ s_4}} \; * \; \underbrace{0}_{\substack{\uparrow\\ s_5}} \; * \; \underbrace{1000}_{\substack{\uparrow\\ s_6}} \; * \; * \; \underbrace{10001}_{\substack{\uparrow\\ s_8}} \; *$$

S_2 is a binary representation of Q defined as follows. A step of Case 1 adds one vertex to G and correspondingly includes one $*$ in Q; similarly, a step of Case 2 adds $q+1$ vertices to G and includes one $*$ and q copies of 0 in Q. Since exactly $n-2$ vertices are added, the total number of these symbols is $n-2$. Each symbol in Q not yet counted corresponds to a vertex covered at the β steps. Since each v_k with $3 \le k \le n-1$ is covered at most once and v_1, v_2, v_n are never covered, the total number of these latter symbols is at most $n-3$. Thus Q has at most $2n-5$ symbols. For the sake of unambiguous decoding, we pad Q with copies of 1 at its end to have exactly $2n-5$ symbols. Since Q uses three distinct symbols, we treat it as an integer of base 3 and convert it to a binary integer. Again, for the sake of unambiguous decoding, we use exactly $\lceil (2n-5)\log 3 \rceil$ bits for this binary integer by padding copies of 0 at its beginning. The resulting binary string is the desired S_2.

For the sake of decoding, we also need to know whether any given $s_{k'}$ is of Cases 1 or 2. Thus, let $S_3 = t_1 \cdots t_\beta$, where $t_{k'} = 1$ if step k' is of Case 1 and $t_{k'} = 0$ otherwise. To save space, note that some bits $t_{k'}$ can be deleted as follows without incurring ambiguity. If step k' is of Case 1a, $t_{k'}$ is deleted because $s_{k'}$ is empty and only a string of Case 1a can be empty. If step k' is of Case 1b, $t_{k'}$ is deleted because $s_{k'}$ starts with 1, while the strings of Case 2 start with 0. If step k' is of Case 1c or 2, $t_{k'}$ remains in S_3. For example, the string S_3 for Figure 3.1 consists of $t_1 = 0$, $t_2 = 0$, $t_4 = 1$, $t_5 = 0$. Thus, S_3 has $\beta_{1c} + \beta_2$ bits, which can be bounded as follows. A step of Case 1 adds one vertex into G and a step of Case 2 adds at least two vertices. Since $n-2$ vertices are added over the β steps, $\beta_{1a} + \beta_{1b} + \beta_{1c} + 2\beta_2 \le n-2$. Since Scheme I assumes $\beta_{1b} \ge \beta_{1c}$, $|S_3| = \beta_{1c} + \beta_2 \le \frac{1}{2}\cdot(\beta_{1b} + \beta_{1c}) + \beta_2 \le \frac{1}{2}\cdot(\beta_{1a} + \beta_{1b} + \beta_{1c} + 2\beta_2) \le 0.5n - 1$.

Since $S = S_1//S_2//S_3$, $|S| \le 2(n-1) + \lceil (2n-5)\log 3 \rceil + 0.5n - 1 \le (2.5 + 2\log 3)n - 9$ bits. This completes the description of the encoding procedure of Scheme I.

Next we describe how to decode S to reconstruct G. This decoding assumes that both S and n are given. Thus, we can uniquely determine S_1, S_2, and S_3. Then we convert S_2 to Q. From Q we can recover all $s_{k'}$ with $1 \le k' \le \beta$. From S_3 and all $s_{k'}$, we can recover all $t_{k'}$ with $1 \le k' \le \beta$. From S_1, we reconstruct T. From T, we find the ordering $v_1, \ldots, v_n$. Afterwards, we draw the edge (v_1, v_2) and perform a loop of steps as follows. Each step is indexed by k' and corresponds to step k' of reconstructing G using the right-most canonical ordering.

If $t_{k'} = 1$, step k' is of Case 1. Thus, a vertex v_k is added at this step, where v_k is the smallest ordered vertex not added into the current graph yet. From T, we can determine the left-most neighbor c_ℓ of v_k in the contour H_{k-1} because c_ℓ is the parent of v_k in T. From $s_{k'}$, we know the number of vertices covered by v_k and hence the right-most neighbor c_r of v_k in the contour H_{k-1}. From $s_{k'}$, we also know which of the covered vertices are connected to v_k. These corresponding edges are added to G.

If $t_{k'} = 0$, step k' is of Case 2. Thus, a chain $v_k, \ldots, v_{k+q}$ is added at this step, where v_k is the smallest ordered vertex not added into the current graph yet. The integer q can be determined from the string $s_{k'}$ by counting its leading copies of 0. From $s_{k'}$, we also know the number of vertices covered at step k', which is the count of 1 in $s_{k'}$. Thus, we know the neighbor c_r of v_{k+q} in the contour H_{k-1}. The chain is added accordingly.

This completes the decoding procedure of Scheme I. It is straightforward to implement the whole Scheme I in $O(n)$ time. If $\beta_{1b} < \beta_{1c}$, we use Scheme II to encode G, which is identical to Scheme I with the following differences. If step k' is of Case 2, $s_{k'}$ consists of q copies of 1 followed by d copies of 0. Also, all bits $t_{k'}$ for steps of Cases 1a and 1c are omitted from S_3 without incurring ambiguity since their corresponding strings $s_{k'}$ are either empty or start with 0, while the strings of Cases 1b and 2 start with 1. We use one extra bit to encode whether we use Scheme I or II. Thus we have the following lemma.

LEMMA 3.2. *Any triconnected plane graph with n vertices can be encoded using at most $(2.5 + 2\log 3)n - 8$ bits. Both encoding and decoding take $O(n)$ time. The decoding procedure assumes that both S and n are given.*

We can improve Lemma 3.2 as follows. Let G^* be the dual of G. G^* has f vertices, m edges and n faces. Since G is triconnected, G^* is also triconnected. Furthermore, if $n > 3$, then $f > 3$ and G^* has no self-loop nor multiple edge. Thus, we can use the coding scheme of Lemma 3.2 to encode G^* with at most $(2.5+2\log 3)f-8$ bits. Since G can be uniquely determined from G^*, to encode G, it suffices to encode G^*. To make S shorter, for the case $n > 3$, if $n \leq f$, we encode G using at most $(2.5+2\log 3)n-8$ bits; otherwise, we encode G^* using at most $(2.5+2\log 3)f-8$ bits. This new encoding has at most $(2.5 + 2\log 3)\min\{n, f\} - 8$ bits. Since $\min\{n, f\} \leq \frac{n+f}{2}$, the bit count is at most $(1.25 + \log 3)m - 2$ by Euler's formula $n + f = m + 2$. For the sake of decoding, we use one extra bit to denote whether we encode G or its dual. Note that if $n = 3$, we can simply encode G using zero bit without ambiguity. Thus we have proved the following theorem.

THEOREM 3.3. *Any triconnected plane graph with n vertices, m edges and f faces can be encoded using at most $(2.5 + 2\log 3)\min\{n, f\} - 7 \leq (1.25 + \log 3)m - 1$ bits. Both encoding and decoding take $O(n)$ time. The decoding procedure assumes that S is given together with n or f as appropriate.*

Remark. There are several ways to improve this coding scheme so that the decoding does not require n as input. One is to use well-known data compression techniques to encode n and append it to the beginning of S using $\log n + O(\log\log n)$ bits [1, 5]. Another is to pad S with copies of 1 at its end so that it has exactly $\lceil(2.5+2\log 3)\min\{n, f\}\rceil-7$ bits. Then, since $2.5+2\log 3 > 1$, given S alone, we can uniquely determine n or f and proceed with the original decoding procedure. With the strings $s_{k'}$, we can unambiguously identify the padded bits.

4. Open problems. This paper leaves several problems open. Since plane triangulations are useful in many application areas, it would be particularly helpful to encode them in $O(n)$ time using close to $1.08m$ bits. Similarly, it would be significant to obtain a linear-time coding scheme for triconnected plane graphs using close to $2m$ bits. Note that Tutte [23] proved an information-theoretic tight bound of $2m + o(m)$ bits for triconnected plane graphs that may contain multiple edges and self-loops. More generally, it would be of interest to encode graphs in a given family in polynomial time using their information-theoretic minimum number of bits. Solving these problems will most likely lead to the discovery of new structural properties of graphs.

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